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A TREATISE ON

ANALYTICAL STATICS

WITH NUMEROUS EXAMPLES

BY

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PREFACE.

IN the first volume of this Treatise the subjects of Attractions, Bending of Rods and Astatics were omitted. These form the substance of this volume, which therefore completes the work.

As so many isolated theorems are included under the head of Attractions, it has been thought necessary to add an Index. While the table of Contents has been made full enough to give a general view of the whole subject, it is hoped that the index will enable the reader to find easily any proposition he is seeking for. Though less necessary, indices to both the other parts of this volume have also been given.

As in the first volume, care has been taken to refer each result to its original author. Many examples also have been taken from the various examination papers set in the University.

I again desire to express my thanks to Mr E. G. Gallop of Gonville and Caius College for the very great assistance he has given me in correcting most of the proof-sheets, and for his many valuable suggestions.

EDWARD J. ROUTH.

PETERHOUSE,
September 3, 1892.

CONTENTS.

ATTRACTIONS.

Introductory Remarks.

Attraction of Rods, Discs, &c.

The Potential.

Spherical Surface.

Laplace's Functions.

Attraction of Ellipsoids.

Rectilinear Figures.

THE BENDING OF RODS.

Introductory Remarks.

ASTATICS.

Astatic Couples.

The Central Ellipsoid.

The Central Plane and the Central Point.

The Confocals.

Arrangement of Poinsot's Central Axes.

Reduction to Three and to Four Forces.

Errata. page 14, line 11, for *frustra* read *frusta*.
page 26, line 7, for r read a. After line 21, for r read r'.

ATTRACTIONS.

Introductory Remarks.

1. **Law of attraction.** If two particles of matter are placed at any sensible distance apart, they attract each other with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance.

2. The reasons for believing the truth of this general law cannot be properly explained until the reader has advanced some way in the study of dynamics. At the same time a large number of theorems which are independent of all dynamical considerations follow from this law. Experience has shown that it is important for the student to acquire an early acquaintance with these results, as he cannot prosecute his studies in the higher dynamics without their assistance. It has therefore been found advantageous to study the attractions of bodies as a part of statics. For this purpose we assume the truth of the law of attraction as a working hypothesis and postpone its verification as a law of nature until the student has read dynamics.

3. Let m, m' be the masses of two particles, r their distance apart; if F be the mutual attraction which each exerts upon the other then F is given by the equation $F = \kappa \dfrac{mm'}{r^2}$.

If f be the acceleration produced by the attraction of m at the distance r, then $f = \kappa \dfrac{m}{r^2}$.

The quantity κ is called the *constant of attraction.* Its magnitude depends on the particular units in which the masses m, m',

the distance r and the force F are measured. To avoid the continual recurrence of this constant running through every equation, it is usual to so choose the units that $\kappa=1$. When this is done the units are called *theoretical* or *astronomical units*.

Putting $\kappa=1$ in the equations, we see that when m and r are both unity the acceleration f is also unity. We infer that the astronomical unit of mass is that mass which, when collected into a particle, produces by its attraction at a unit of distance the unit of acceleration.

The expression for F shows that the unit of force is the attraction which a particle whose mass is the astronomical unit of mass exerts on an equal particle at a unit of distance.

To avoid the continual repetition of the same set of words, we shall use the phrase attraction at a point to mean the attraction on a unit of mass collected into a particle and placed at that point.

4. It is convenient to use different systems of units for different purposes. The astronomical units should be used in analytical investigations. In any numerical applications we may choose such units of space and time as we may find convenient, and then introduce into our formulæ the appropriate value of κ.

It may be noticed that in using different units for different purposes we are following the analogy of other mathematical sciences. In practical trigonometry we measure angles in degrees, in theoretical trigonometry we adopt that unit by which our analytical formulæ are most simplified. So also in algebra we have one base in logarithms for use in calculations and another for theoretical investigations; and so on through all the sciences.

5. **Numerical estimate.** To obtain a numerical estimate of the magnitude of the force of attraction, we must determine by experiment the mutual attraction of some two bodies. We may exhibit the result in either of two forms: (1) we may determine the value of κ when the units of space, mass, &c. have been chosen; (2) we may determine the magnitude of the astronomical unit of mass by expressing it as a multiple of some other known mass.

The two bodies on which the experiment should be tried are obviously the earth and some body at its surface. Regarding the earth as a sphere, whose strata of equal density are concentric spheres, it will be shown further on that its attraction on all external bodies is the same as if its whole mass were collected into a particle and placed at its centre. If then m be the mass of the earth, a its radius, its accelerating attraction on a body at its surface is $\kappa m/a^2$. Let g be the acceleration actually produced by the attraction of the earth on any body placed at its surface. We thus form the equation $\frac{\kappa m}{a^2}=g$.

Several experiments have been made to determine the mean density of the

earth. One of these is the Cavendish experiment, but there have been others conducted on different plans. The result is that the mean density has been variously estimated to be from $5\frac{1}{2}$ to 6 times that of water. According to Baily's repetition of the Cavendish experiment the ratio is 5·67. Representing this ratio by β, we learn that the attraction of a sphere of water, of the same size as that of the earth, will produce in a body, placed at its surface, an acceleration equal to g/β.

Taking the equation $F=\kappa\frac{mm'}{r^2}$, let us find the value of κ in the C. G. S. system of units. Let m represent the mass of the earth, then since the mass of a cubic centimetre of water is one gramme nearly, we have $m=\frac{4}{3}\pi a^3\beta$ grammes. Taking $\beta=5\cdot67$, this gives $m=6\cdot14\times10^{27}$ grammes nearly. Let the attracted mass be one gramme, and let $r=a$. In the C. G. S. system the unit of force is the dyne, and $m'g$ when so measured is 981 dynes. Substituting $F=m'g$, $a=6\cdot371\times10^8$, we find

$$\kappa=\frac{6\cdot48}{10^8}=\frac{1}{1\cdot543\times10^7}.$$

If therefore the attracting masses are measured in grammes and the distances in centimetres, the expression for F with this value of κ gives the attraction in dynes. See Vol. I. Art. 11.

To find the astronomical unit of mass when the centimetre and the second are the other units, we refer to the equation $f=\kappa\frac{m}{r^2}$. Let m be the mass measured in grammes which by its attraction at the distance of a centimetre produces unit acceleration. Then putting $f=1$, $r=1$, we have $m=1/\kappa$. The astronomical unit is therefore $1\cdot543\times10^7$ grammes.

Let us next find the value of κ when the system of units is that based on the foot, pound and second (see Vol. I. Art. 11). Let m be the mass of the earth, then since a cubic foot of water weighs 61 lbs. nearly, we have m equal to the mass of $\frac{4}{3}\pi a^3\beta$. 61 lbs. Let m' be one pound, then $F=32\cdot18$ poundals. We therefore have

$$\kappa=\frac{\cdot1}{9\cdot3\times10^8}.$$

If the attracting masses are measured in pounds and the distances in feet, the expression for F with this value of κ gives the attraction in poundals.

To find the astronomical unit of mass when the foot and second are the units of space and time we use the equation $f=\frac{\kappa m}{r^2}$. Let m be the mass measured in pounds which by its attraction at the distance of a foot produces the unit of acceleration. Then putting $f=1$, $r=1$, we have $m=1/\kappa$. The astronomical unit is therefore $9\cdot3\times10^8$ pounds.

6. Ex. 1. Show that the mass which at the distance of one centimetre from an equal mass attracts it with the force of one dyne is 3928 grammes. *Everett's Units and Physical Constants.*

Ex. 2. Show that a cubic foot of water, collected into a particle, attracts an equal particle at the distance of one foot with a force equal to the weight of $\frac{1}{1\times10^6}$ pounds.

7. **Law of the direct distance.** There are other laws besides that of the inverse square which may govern the attraction of bodies in special cases. Some of these will be mentioned as we

proceed. But the most useful is that in which the attraction varies as the distance. In this case the attraction of two particles, each on the other, is represented by $F = mm'r$, where m, m' are their masses, and r, the distance between them.

8. When the attraction obeys the law of the direct distance, the resultant attraction of any body at any point is found at once by using Art. 51 of Vol. I. Let O be any point, A_1, A_2, &c. the positions of the attracting particles; let m_1, m_2, &c. be their masses. The component attractions at O are then given by $X = \Sigma mx = \bar{x}\Sigma m$, $Y = \bar{y}\Sigma m$, $Z = \bar{z}\Sigma m$, where $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coordinates of the centre of gravity of the body or system of attracting points.

It immediately follows that the resultant attraction at O is the same as if the whole mass Σm of the attracting system were collected into a single particle placed at the centre of gravity. *The resultant force on a particle at O tends therefore towards the centre of gravity of the attracting system, and is proportional to the distance of the attracted point from it.*

9. In what follows, when no special law of force is mentioned, it is to be understood that the law meant is that of the inverse square. This is often called the Newtonian law.

When the law of attraction is said to be $f(r)$, it is meant that the mutual attraction of two particles whose masses are m, m' placed at a distance apart equal to r is $mm'f(r)$.

Attraction of rods, discs, &c.

10. **Attraction of a rod.** *To find the attraction of a uniform thin rod AB at any external point P.*

Let m be the mass of a unit of length of the rod, p the length of the perpendicular PN from P on the rod. Let QQ' be any element of the rod, $NQ = x$; let also the angle $NPQ = \theta$, then $x = p \tan\theta$.

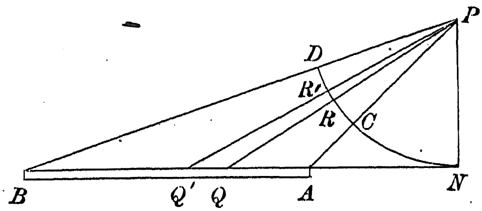
The attraction at P of the element QQ' is

$$\frac{m\,dx}{PQ^2} = \frac{m\,d(p\tan\theta)}{(p\sec\theta)^2} = \frac{m\,d\theta}{p}.$$

Let X, Y be the resolved attractions at P parallel and perpendicular to the length AB. Let the angles NPA, NPB be α, β, then

$$X = \int m\frac{d\theta}{p}\sin\theta = \frac{m}{p}(\cos\alpha - \cos\beta) \quad \ldots\ldots \ldots\ldots (1),$$

$$Y = \int m \frac{d\theta}{p} \cos\theta = \frac{m}{p} (\sin\beta - \sin\alpha) \text{ .... (2).}$$



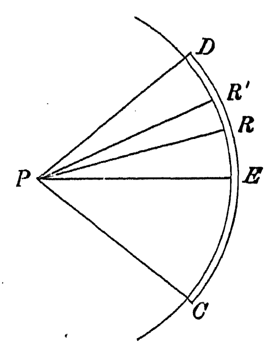
11. Substituting for $\cos\alpha$, $\cos\beta$ their values obtained from the triangles PNA, PNB, the resolved attraction parallel to the rod takes the useful form $$X = \frac{m}{PA} - \frac{m}{PB} \text{ ....(3).}$$

It should be noticed that this is the attraction at P of the rod AB resolved in the direction from A towards B.

12. Describe a circle with centre P and radius PN and let the portion CD included between the distances PA, PB represent a thin circular rod of the same material and section as the given rod AB.

The attraction at P of the element RR' of the circular rod is therefore $\dfrac{m \,.\, RR'}{PR^2} = m\dfrac{pd\theta}{p^2} = m\dfrac{d\theta}{p}$. But this has just been proved to be the same as the attraction of the element QQ'. Thus each element of the rod AB attracts P with the same force as the corresponding element of the rod CD. *The resultant attraction of the straight rod AB is therefore the same in direction and magnitude as that of the circular rod CD.*

13. The resultant attraction at P of the circular rod CD must clearly bisect the angle CPD. It immediately follows that *the direction of the resultant attraction at P of a straight rod AB bisects the angle APB.*



To find the magnitude of the resultant attraction at P of the circular arc CD, we draw PE bisecting the angle CPD. Let the angle any radius PR makes with PE be ψ. Let 2γ be the angle CPD. Since $RR' = pd\psi$ the attraction of the whole circular arc when resolved along PE is

$\int \frac{mpd\psi}{p^2} \cos\psi = \frac{m}{p} . 2 \sin\gamma$, the limits of the integral being $\psi = -$ and $\psi = \gamma$. *The magnitude F of the resultant attraction at P c a straight rod AB is given by* $F = \frac{2m}{p} \sin \frac{APB}{2}$.

14. When the rod AB is infinite in both directions the angl APB is equal to two right angles. *The resultant attraction of a infinite rod at any point P is equal to $\frac{2m}{p}$, and it acts along th direction of the perpendicular p drawn from P on the rod.*

15. Ex. 1. Deduce from the expressions for the components X, Y of th attraction given in Art. 10 the direction and magnitude of the resultant attraction and show that the results agree with those given in Art. 13.

Ex. 2. When the point P lies in the straight line AB the expressions for X anc Y take a singular form. Show that the attraction parallel to the rod is still giver by the form (3) which is free from singularity, and that the expression (2) for the attraction perpendicular to the rod reduces to the value zero.

To prove the former result, measure two equal lengths PC, PD from P in opposite directions. The x attraction of CD on P is clearly zero, while the x attractions of AC and DB are each given by the form (3) and when added togethei give the attraction of the whole rod.

Ex. 3. If two forces be applied at P acting along AP, PB taken in order, and each equal to $\frac{m}{p}$, prove that their resultant is equal in magnitude to the attraction of the rod AB and acts in a direction perpendicular to that attraction.

Ex. 4. The attraction of the straight rod AB at a point P is the resultant of two forces each equal to f acting at P towards the extremities of the rod where

$$f = \frac{2m . AB}{(AP+BP)^2 - AB^2}.$$

Show also that if P move on an ellipse whose foci are the extremities of the rod, the magnitude of each force f is constant and equal to $\frac{m . AB}{2b^2}$, where $2b$ is the minor axis of the ellipse.

Ex. 5. The sides of a triangle are formed of three thin uniform rods of equal density. Prove that a particle attracted by the sides is in equilibrium if placed at the centre of the inscribed circle.

If one side of the triangle repel while the other two attract the particle, prove that the centre of an escribed circle is a position of equilibrium.

[Math. Tripos.]

This follows at once from Art. 12. Draw straight lines from the centre I of the inscribed circle to the corners A, B, C of the triangle cutting the circle in A', B', C'. The attractions of the sides AB, BC, CA are the same as those of the arcs $A'B'$, $B'C'$, $C'A'$, that is their resultant attraction is the same as that of the whole circle on the centre. This attraction is clearly zero.

Ex. 6. Four uniform straight rods of equal density form a quadrilateral, and their lengths are such that the sum of two opposite sides is equal to the sum of the

other two opposite sides. Find the position of equilibrium of a particle under the attraction of the four sides.

Ex. 7. Every particle of three similar uniform rods of infinite length, lying in the same plane, attracts with a force varying inversely as the square of the distance; prove that a particle, subject to the attraction of the rods, will be in equilibrium if it be placed at the centre of gravity of the triangle enclosed by the rods.

[Math. Tripos, 1859.]

Ex. 8. Two uniform rods occupy the positions of two tangents TP, TQ to a parabola: show that the directions of their resultant attractions at the focus S make equal angles with ST, and that their magnitudes are as $\sqrt{SQ}$ to $\sqrt{SP}$.

Ex. 9. A particle is placed at any point P on the bisector of the angle C of a triangle. Show that the direction of the resultant attraction of the three sides at P bisects the angle APB and is equal in magnitude to $2m\left(\frac{1}{\gamma}-\frac{1}{\alpha}\right)\sin\frac{APB}{2}$, where α and γ are the perpendiculars from P on the sides BC, AB respectively.

Ex. 10. A particle is placed at any point P under the attraction of the sides of the triangle ABC. If α, β, γ be the trilinear coordinates of P referred to ABC, prove that the resultant attraction at P is equal in magnitude to the resultant of the three forces $m\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)$, $m\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)$, $m\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)$ acting along AP, BP, CP respectively and its direction is perpendicular to the resultant of those forces.

Ex. 11. Two uniform parallel straight rods AB, CD attract each other: show that the components of their mutual attraction respectively perpendicular and parallel to the rods are

$$Y=\frac{2mm'}{p}(BC-BD-AC+AD),$$
$$X=2mm'\log\frac{BC'+BC}{AC'+AC}\cdot\frac{AD'+AD}{BD'+BD},$$

where C', D' are the projections of C, D on the rod AB, p the distance between the rods, and m, m' the masses per unit of length.

Ex. 12. P is a particle in the diagonal AC of a square $ABCD$, and within the square; show that the attraction of the perimeter of the square upon P is equal to $M\cdot\frac{OP}{PA\,.\,PB\,.\,PC}$; where M is the mass of the perimeter, O the centre of the square.

[Trin. Coll., 1882.]

Ex. 13. The faces of a rectangular parallelepiped are covered uniformly with attracting matter. Prove that the difference of the attractions at an internal point of two opposite faces resolved perpendicular to two other faces is equal to the difference of the attractions of these second two faces resolved perpendicular to the first two. [Trin. Coll., 1881.]

Ex. 14. A uniform wire of infinite length attracts according to the inverse nth power of the distance: show that the resultant attraction is $\sqrt{\pi}\,\frac{m}{c^{n-1}}\cdot\frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n+\frac{1}{2})}$, where m is the mass per unit length of the wire and c its least distance from the attracted point. [St John's Coll., 1884.]

Ex. 15. Show that gravity is diminished by $\frac{\pi+2\log 2}{\pi}\,\frac{3\sigma}{4\rho}\,\frac{a}{r}$ of itself at the middle point of a canal of rectangular section, whose length is great compared

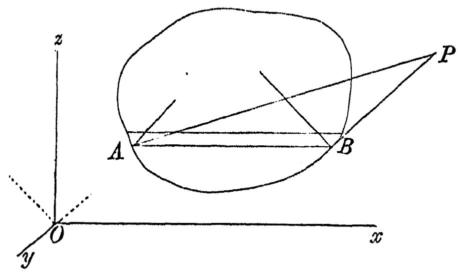
with its depth, where a is the depth, $2a$ the breadth, r the radius of the ea supposed spherical, and $1-\sigma/\rho$ the ratio of the density of water to the mean dens of the earth. [June Exam., 188

Ex. 16. A mountain is in the form of an infinitely long wedge formed by t planes meeting along a line whose height above the earth's mean surface is unifor If the transverse vertical section at any point P on the side of the mountain b triangle ABC, then the whole horizontal attraction of the mountain at P will vary

$$h(\log\sin B-\log\sin\alpha)+k\{(B+\alpha)\sin C+\cos C\log\sin A-\cos C\log\sin(C-\alpha)\},$$

where h and k are the lengths of the perpendiculars drawn from P to the base B and to the opposite face AC respectively, where α is the angle PCB. [Math. Tripo

16. **Gauss' theorem.** The attraction at P of a solid body of any form wh



resolved parallel to any straight line (taken as the axis of x) is given by

$$X=\rho\int\frac{\cos\phi}{r}\,d\sigma,$$

where ρ is the density of the body, r the distance from P of any element $d\sigma$ of th area of the surface, and ϕ is the angle the normal at $d\sigma$ drawn inwards makes wit the positive direction of the axis of x.

To prove this, we divide the body into elementary columns or rods whose length are parallel to the axis of x. Let AB be one of these. Since the area of a sectio of the rod is $dydz$, the mass m per unit of length is $\rho dydz$. The resolved attractio of the column is therefore equal to $\rho dydz\left(\frac{1}{PA}-\frac{1}{PB}\right)$.

Let $d\sigma$, $d\sigma'$ be the elementary areas intercepted on the surface of the solid b the sides of the column AB; ϕ and ϕ' the angles the normals at A and B, *drawn inwards*, make with the axis of x. By drawing parallels to these normals at th origin, we see that the first of these angles is acute in the figure and the secon obtuse. We have therefore $dydz=d\sigma\cos\phi=-d\sigma'\cos\phi'$.

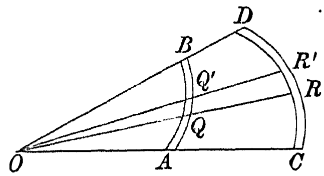
The x attraction of the column is therefore equal to $\rho\left(\frac{\cos\phi}{PA}d\sigma+\frac{\cos\phi'}{PB}d\sigma'\right)$. Writing r for PA or PB and integrating for all the columns, the result follows a once. By Art. 15, Ex. 2, the theorem is also true when P is internal.

17. **Curvilinear rods.** The method by which the attraction of the straight rod AB is replaced by that of the circular arc CD in Art. 12 may be extended to other curves.

Two curvilinear rods AB, CD are so related that if any two radii vectores OAC, OBD are drawn, the attractions of the inter-

cepted arcs AB, CD at the origin O are the same in direction and magnitude. It is required to find the relation between the densities of the rods.

Since the attractions are equal for all arcs, they are equal for infinitesimal arcs. Let OQR, $OQ'R'$ be two consecutive radii vectores; ds, ds' the arcs QQ', RR'; m, m' the masses at Q, R per unit of length. Then if the law of attraction is the inverse nth power of the distance we have



$$\frac{mds}{r^n} = \frac{m'ds'}{r'^n},$$

where $r = OQ$, $r' = OR$. If ϕ, ϕ' be the angles the radius vector OQR makes with the tangents at Q and R, this gives

$$\frac{m}{r^{n-1}\sin\phi} = \frac{m'}{r'^{n-1}\sin\phi'} \quad \text{.....................(1).}$$

The densities of the curvilinear rods at corresponding points must therefore be proportional to $r^{n-1}\sin\phi$.

18. If the two curves are so related that *each is the inverse of the other*, we have $OQ \cdot OR = OQ' \cdot OR'$. A circle can therefore be described about the quadrilateral $QRR'Q'$. In the limit when QQ', RR' become tangents this gives $\sin\phi = \sin\phi'$. If also $n=1$, we see that $m = m'$. It follows therefore that when the law of attraction is the inverse distance, any curvilinear rod and its inverse, if of equal uniform density, equally attract the origin.

19. Ex. 1. If the law of attraction be the inverse square, two curvilinear rods in one plane equally attract the origin, if the densities at corresponding points in the two rods are proportional to the perpendiculars from the origin on the tangents.

Ex. 2. Let the law of attraction be the inverse distance and let P be any point attracted by a uniform rod AB. Draw PN perpendicular to the rod and describe a circle on PN as diameter. Prove that the attraction of AB at P is the same as that of the corresponding arc CD of the circle intercepted between the straight lines PA, PB, if the line densities are equal.

Ex. 3. A uniform homogeneous wire PAP', of which A is the middle point, is bent into the form of an arc of a loop of the lemniscate of which A becomes the vertex: prove that the resultant attraction on the wire, arising from a centre of force at the node O, attracting according to the law of the inverse square, varies as

$$\left(\frac{1}{OP^2} - \frac{1}{OA^2}\right)^{\frac{1}{2}}.$$

[Math. Tripos, 1857.]

Ex. 4. Two rigid and equal semicircular arcs of matter with uniform section and density are hinged together at both extremities. The matter attracts according to the law of gravitation. If equal and opposite forces applied along the line joining the middle points of the semicircles keep them apart with their planes at right angles, the magnitude of each force will be $4m^2\log(1+\sqrt{2})$, where m is the mass of unit length of arc. [Math. Tripos, 1874.]

20. Some inverse problems. Ex. 1. A uniform rod is bent into the form a curve such that the direction of the attraction of *any* arc PQ at the origin bisects the angle POQ. Show that the curve is either a straight line or a cir whose centre is O.

The data lead to the differential equation $\int \frac{ds}{r^2} \sin \theta = \tan \frac{\theta}{2} \int \frac{ds}{r^2} \cos \theta$. T limits of the integrals being 0 and θ. The equation may be solved by diff entiation. put $\frac{ds}{r^2}$ = P & dif with respect to θ 2 t

Ex. 2. Find the law of density of a curvilinear rod of given form in order th the attraction at O of any arc PQ may bisect the angle POQ. The result is th the density must be proportional to the perpendicular on the tangent.

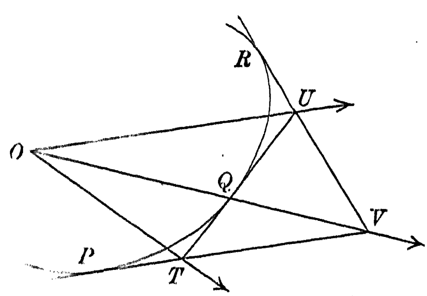
This follows from the last example by Art. 19, Ex. 1.

Ex. 3. A uniform rod is bent into the form of a *given curve*, find the law attraction in order that the direction of the attraction at O of any arc PQ m bisect the angle POQ. The result is that the law of attraction is $f(r) = Ap/r^2$, whe A is a constant and p is the perpendicular on the tangent and is given as a functi of r when the equation to the curve is known.

Ex. 4. A uniform rod is bent into a curve such that the direction of t attraction at the origin of any arc PQ passes through the centre of gravity of t arc. Prove that either the law of attraction is the direct distance or the cur is a straight line which passes through the origin.

Ex. 5. If any uniform arc of an equiangular spiral attract a particle placed the pole, prove that the resultant attraction acts along the line joining the pole t the intersection of the tangents at the extremities of the arc.

Prove also that if any other given curve possess this same property, the la of attraction must be $F = \frac{\mu}{p^2} \frac{dp}{dr}$, where p is the perpendicular drawn from the attracted particle on the tangent at the point of which the radius vector is r.



Reversing the attracting forces, we may regard the rod as acted on by a centre of repulsive force. Since the resultant force on any arc PQ acts along OT, where T is the intersection of the tangents at P and Q, we may resolve that force into two components which act along TP and TQ. It follows that th resultant force on any arc PQ may be balanced by two forces or tensions actir along the tangents at P and Q.

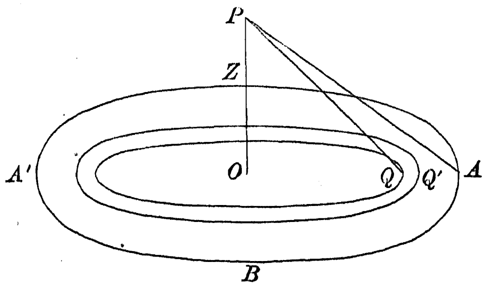
To complete the analogy of the force at P to a tension, we must show that th force is always the same whatever the length of the arc PQ may be. To prov this let PQ, QR be two contiguous arcs, and let the tangents at P, Q meet in those at Q, R in U, those at P, R in V. Resolving the forces at T, U, V as befor the components along PT, QT and RU, QU must together be equivalent to th components along PV, RV. We have to deduce from this that the component along PT and PV are equal. This follows at once by taking moments about U.

The conditions of equilibrium of the rod are therefore the same as those a string acted on by a central force. Referring to Art. 474, Vol. I., the tension

obviously $T=A/p$ and the force $f(r)$ has the value given above. See the *Solutions of the Senate House problems for the year* 1860, page 61. The analytical solution leads to an interesting differential equation which can be solved without great difficulty.

21. **Attraction of a circular disc.** *To find the attraction of a uniform thin circular disc at any point in its axis.*

Let O be the centre, ABA' the disc seen in perspective; OZ the axis, i.e. a straight line drawn through O perpendicular to the plane of the disc. Let a be the radius of the disc, m the mass per unit of area. Let P be the point at which the attraction is required, $OP=p$, and the angle $OPA=\alpha$.



Describe an elementary annulus represented in the figure by QQ'. Let x, $x+dx$ be its radii, and let θ be the angle OPQ. The resultant attraction of the disc at P is

$$F=\int\frac{2\pi x dx\,.\,m}{QP^2}\cos\theta,$$

where the limits of the integral are $x=0$ and $x=a$. Since $x=p\tan\theta$ and $QP=p\sec\theta$, we find

$$F=2\pi m\int\sin\theta d\theta=2\pi m\,(1-\cos\alpha).$$

Here α is the *acute angle* subtended at the attracted point by the radius of the disc.

22. From this we deduce the attraction of an infinite plate or disc by putting $\alpha=\frac{1}{2}\pi$. We thus find that *the attraction of an infinite plate is* $2\pi m$.

It appears from this that the attraction of an infinite plate is independent of the distance of the particle attracted from the plate. At first sight this result may appear anomalous, but we may understand how it can happen by considering what elements of the disc are effective in producing the attraction. Each element of an annulus QQ', whose centre is O, attracts P with a force acting along the straight line joining P to that element, and the component of force along PO is obtained by multiplying this attraction by cos OPQ. When the point P is near O, this cosine is small and therefore it is only the portion of the disc near O which exerts any

sensible attraction in the direction PO. As P recedes from O, the cosine for ‹ annulus gets larger and the resolved attraction becomes greater. Thus the are effective attraction increases in size as the particle recedes. At the same tim the particle P recedes from O the actual attraction of each annulus on it decrea It follows from the analysis in the last article that the increase of effective ; just balances the decrease of attraction due to increased distance, so that on whole the attraction is independent of the distance.

Ex. 1. If g, g' be the attractions due to gravity on two tablelands wl difference of level is x, show that $g'=g\left(1-\frac{5}{4}\frac{x}{a}\right)$ approximately, where a is radius of the earth.

To obtain this result, we regard the attraction of the tableland as sensibly same as that of an infinite plate, Art. 22. The attraction is therefore $2\pi\rho x$, wl ρ is the density of the tableland or flat mountain. If ρ' be the mean densit the earth, its attraction viz. g is $\frac{4}{3}\pi\rho' a$. There are reasons for believing that mean surface density of the earth is about half the mean density of the wl earth; when therefore the true density of the tableland is unknown we may as approximation put $\rho=\frac{1}{2}\rho'$. The attraction of the tableland is thus approxima $\frac{3}{4}gx/a$. The attraction of the earth at the altitude x is

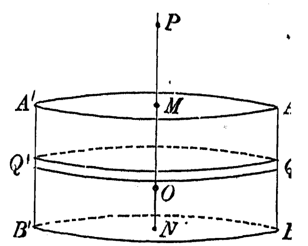
$$g\left(\frac{a}{a+x}\right)^2=g\left(1-2\frac{x}{a}\right)$$

approximately. Adding this to the attraction of the tableland we arrive at result given.

This theorem was first used by Bouguer in his *Figure de la Terre*. A sh account of this treatise is given in Art. 363 of Todhunter's *History of Attracti* &c. A similar result is also given by Poisson in Art. 629 of his *Traité Mécanique*. It is often called Dr Young's rule.

Ex. 2. Dr Siemens invented an instrument to measure the depth of the under a ship on the principle of balancing gravitation by the force of a spring. the mean surface density of the earth be three times that of sea water, and mean density of the whole earth five and a half times that of sea water, show t for a depth h of sea the diminution of gravity is $\frac{9}{11}hg/R$, where R is the radius the earth.

23. **Attraction of a Cylinder.** Ex. 1. Find the attraction of a uniform sc right circular cylinder at a point P on its axis.



Let ρ be the density of the cylinder, a its radius. Let O be the centre of gravity, $OP=p$. Let us take as the element of volume the slice of the cylinder between two planes drawn perpendicular to the axis at distances x and $x+dx$ from O.

First, let P be outside the cylinder. Let 2θ be the angle subtended at P by any diameter QQ' of the slice, and let $PQ=r$. Since the mass per unit of area of the slice is $m=\rho dx$, the attraction at P is $2\pi\rho dx\,(1-\cos\theta)=2\pi\rho dx\left(1-\frac{p-x}{r}\right)$. But $(p-x)^2+a^2=r^2$, $\therefore (x-p)\,dx=r$ The whole attraction of the cylinder at P is therefore $F=2\pi\rho\int(dx+dr)$, where t limits of integration are $x=-\frac{1}{2}AB$ to $x=\frac{1}{2}AB$ and $r=PB$ to $r=PA$. The resulti

attraction is therefore $F=2\pi\rho(AB+PA-PB)$, where AB is any generating line and A is the extremity nearest to P. We notice that AB is equal to the difference of the distances from the plane sections passing through A and B.

Next, let P be inside the cylinder, but nearer to the plane section $A'A$ than to $B'B$. Since θ is the *acute angle* subtended at P by the radius of the attracting slice, we must equate $\cos\theta$ to $\pm(p-x)/r$, the sign being different on opposite sides of P. To avoid this discontinuity we draw a plane $C'C$ perpendicular to the axis so that P lies midway between the sections $A'A$ and $C'C$. The resultant attraction of the matter between $A'A$ and $C'C$ at P is therefore zero. The resultant attraction of the rest of the cylinder is given by

$$F=2\pi\rho(CB+PC-PB)$$
$$=2\pi\rho(CB+PA-PB).$$

Here CB is equal to the difference of the distances of P from the plane sections through A and B, measured positively in opposite directions.

Another Solution. We may also find the attraction by dividing the cylinder into elementary columns or filaments parallel to the axis. This would be repeating the proof of Gauss' theorem given in Art. 16.

To apply that theorem, we notice that $\cos\phi=0$ at all points on the convex surface and $\cos\phi=1$ at all points on either of the plane faces, where ϕ is the angle made by the normal to the surface and the axis. The resolved force parallel to the axis is therefore the difference between the values of the integral $\int\rho\dfrac{d\sigma}{r}$ for the two plane faces, where r is here the distance of $d\sigma$ from P. Since $d\sigma=2\pi r dr$, and the limiting values of r for the faces AA', BB' respectively are PM to PA and PN to PB we easily arrive at the same result as before.

Ex. 2. Find the ratio of the radius of the base to the height of a right circular cylinder of given volume so that the attraction at the centre of one of the circular ends may be the greatest possible. The required ratio is $\frac{1}{4}(9-\sqrt{17})$. Playfair's problem. See Todhunter's *History*, 1585.

Ex. 3. A right circular cylinder is of infinite length in one direction and is homogeneous. If the finite extremity be cut off perpendicularly to the generators, prove that the attraction on a unit particle placed at the centre of this end is $2M/a$, where M is the mass per unit of length.

If the cylinder be elliptic, of the same density and mass per unit of length as before, and of eccentricity e, then the attraction will be n times the former value,

where $n=\dfrac{2}{\pi}(1-e^2)^{\frac{1}{4}}\displaystyle\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-e^2\sin^2\theta}}$. [St John's Coll., 1887.]

Ex. 4. A solid right circular cylinder of uniform density ρ stands on the plane of xy and is infinite in the positive direction of the axis of z. Show that the z component of its attraction at a point P of its base is ρl, where l is the perimeter of an ellipse having the base for the auxiliary circle and P for one focus. See Art. 11.

Ex. 5. A vertical solid cylinder of height h, radius a, and density ρ, bounded by plane ends perpendicular to the axis, is divided by a plane through the axis into two parts. Show that the horizontal attraction of either part at the centre of the base is

$$2h\rho\log\frac{a+\sqrt{(a^2+h^2)}}{h}.$$ [Coll. Ex., 1888.]

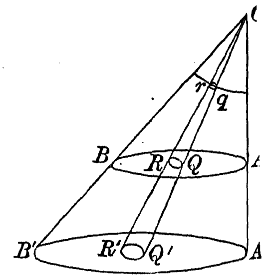
Ex. 6. The tide in the bay of Fundy rises 100 feet from low to high water mark. It has been proposed to find the density of the earth by determining the attraction of the tide-wave on a plumb-line at high and low tide on the same principle as

Maskelyne's experiment at Schehallien. Supposing the attraction of the tide-w at a point O on the shore to be represented by that of the water within a cylin whose axis is the vertical at O, whose height l is 100 feet and radius r, show t the deviation of the plumb-line is $\frac{3l}{2\pi RD}\log\frac{2r}{l}$, where R is the radius of the ea D its mean density, and r is large compared with l.

Show that this expression increases slowly compared with r, and that if r taken between 2 and 4 miles the deviation to be observed will be about one-fifth a second. This is much smaller than the deviation to be observed in Maskely experiment which was about eleven seconds. On the other hand the attracting m is a homogeneous body instead of a heterogeneous mountain.

24. **Attraction of a surface.** *All frustra of a unifo cone which are of the same thickness, and have their plane fa parallel to a given plane, exert equal attractions at the vertex.*

Let AB, $A'B'$ be two thin parallel laminæ of the sa thickness dt. Let ρ be the density of the cone. With the same vertex O describe an elementary cone cutting the laminæ in QR, $Q'R'$. The attractions of QR, $Q'R'$ at O are to each other as their masses divided by the squares of the distances. Since the thicknesses are equal, the masses are proportional to the areas, and these by similar figures are proportional to the squares of the distances OQ, OQ'. Thus t attractions of the elements QR, $Q'R'$ at O are equal. Hence t attractions of the laminæ AB, $A'B'$ at O are the same both direction and magnitude.



This being true for all thin laminæ must, by integration, also true for all thick sections. And in general any two parall slices of the same cone, whether thick or thin, attract the vertex the same direction with forces proportional to their thicknesses.

25. As the attraction of the element QR at any point O wanted in several theorems further on, it is convenient to determi an expression for its magnitude.

Let $d\sigma$ be the area of the element QR, m its mass per unit area, r its distance from O; the attraction at O is then $\frac{md\sigma}{r^2}$.

To simplify this expression, we use the solid angle subtende at O by the area. Just as in plane trigonometry an angle measured by the arc subtended in a circle of unit radius, so th solid angle contained by any cone is measured by the surface cu

off by the cone from a sphere of unit radius with its centre at the vertex.

Let the elementary cone whose base is QR intercept on the unit sphere an elementary area qr, and let this area be $d\omega$, then $d\omega$ measures the solid angle subtended at O. Let ψ be the angle the normal to the elementary area QR makes with the radius vector OQ, then $d\sigma \cos\psi$ is the area of a section of the cone made by a plane drawn through Q perpendicular to OQ. Hence by similar figures

$$\frac{d\sigma \cos\psi}{r^2} = \text{area } qr = d\omega.$$

The attraction of the element is therefore $m \sec\psi \,.\, d\omega$. If p be the perpendicular from O on the plane of the element, then $r\cos\psi = p$, and the attraction of the element at O may also be written in the form $m\dfrac{rd\omega}{p}$.

26. It follows from this result that the attraction at P of an element $d\sigma$ when resolved perpendicular to its plane is $md\omega$.

Hence we may deduce by integration that *the attraction at P of a plane uniform lamina of any form when resolved perpendicular to the plane is $m\omega$, where m is the mass of a unit of area of the lamina, and ω is the solid angle subtended at P by the lamina.* This theorem is due to Playfair.

Ex. If l, m, n be the direction cosines of the radius vector of an element of a surface, and if l, m, n can be expressed in terms of two parameters a and b, show that the normal attraction of the element on the origin is $\Delta da\,db\,dk$, where dk is the thickness of the element and Δ is the determinant in the margin.

$$\left| \begin{array}{ccc} l, & m, & n \\ \dfrac{dl}{da}, & \dfrac{dm}{da}, & \dfrac{dn}{da} \\ \dfrac{dl}{db}, & \dfrac{dm}{db}, & \dfrac{dn}{db} \end{array} \right|$$

[Caius Coll.]

27. The method explained in Art. 17 by which the attraction at the origin of one thin rod may be replaced by that of another of a more convenient form may be extended to surfaces.

Let the law of attraction be the inverse nth power of the distance. Referring to the figure of Art. 17 and equating the attractions of the elementary areas QR, $Q'R'$, we have

$$\frac{md\sigma}{r^n} = \frac{m'd\sigma'}{r'^n}.$$

Since by Art. 25 $d\sigma\cos\psi = r^2 d\omega$, this gives

$$\frac{m}{r^{n-2}\cos\psi} = \frac{m'}{r'^{n-2}\cos\psi'}.$$

It follows that *if two curvilinear laminæ are so related that*

their masses per unit of area at points on the same radius vect drawn from a point O are connected by the above equation, then t attractions at O of the portions included within any conical surfa whose vertex is O are the same in direction and magnitude.

For example, if the law of attraction be the inverse cube, tl attraction at a point O of any portion of a thin plane area is tl same in direction and magnitude as that of the correspondir portion of a spherical surface having its centre at O, and touchir the plane, the masses per unit of area of the plane and sphe being equal. This exactly corresponds to the theorem in Art. 1 which connects the attraction of a straight rod with that of circle.

If the plane area be bounded by a conic, its attraction at an point O acts along the axis of the enveloping cone whose verte is O.

28. Ex. 1. Show that the attraction at a point O of any portion of a thi plane disc is the same in direction and magnitude as that of the correspondir portion of a spherical surface having for a diameter the perpendicular ON draw from O on the plane. The two attracting surfaces are supposed to be homogeneo and of equal mass per unit of area.

Ex. 2. A tetrahedron is constructed of thin metal: prove that a particle und its attraction would remain at rest if placed at the centre of the inscribed spher provided the small thickness of the faces at any point be inversely proportional t the distance from that centre. [Math. Tripos, 1880

Ex. 3. A thin uniform lamina lies in the plane of xy and is bounded by tl focal conic $\dfrac{x^2}{a^2-c^2}+\dfrac{y^2}{b^2-c^2}=1$. If the law of attraction be the inverse cube of tl distance, show that the attraction of the lamina at any external point P acts alon the normal to the confocal conicoid which passes through P. [Coll. Exam

29. **The solid of greatest attraction.** *To find the solid of revolution of give mass which exerts the greatest attraction at a point O situated on the axis.*

Let us trace the surface such that the attraction at the given point O of a partic of given mass m placed at any point of the surface when resolved along the given ax is equal to a given constant C. Taking O for origin and the given axis as the axi of reference, the equation of that surface is clearly $\dfrac{m}{r^2}\cos\theta=C$. By giving C differer values we obtain a system of surfaces. It is evident from the definition that th surface defined by any value of C lies outside that defined by a greater value of C It follows that the resolved attraction of a particle lying on any surface is greate than that of an equal particle situated on any external surface.

It is evident from the equation that all these surfaces are similar and similarl situated, and that they all touch a plane drawn through O perpendicular to th given axis.

Let us select that surface whose volume would just contain the given mass The solid of greatest attraction must coincide with the surface thus selected; for i

any portion lies outside the selected surface, the attraction would be increased by moving that portion into the vacant places within the selected surface and thus filling them up.

The solid of greatest attraction has therefore such a form that the attraction at the given point of a given particle placed at any point of the surface when resolved along the given axis is always the same.

The problem of finding the solid of greatest attraction was proposed and solved by Silvabelle. The principle used above, that the resolved attraction must be constant over the surface, is due to Playfair. The following example is also due to him.

30. Ex. Supposing the law of attraction to be the inverse nth power of the distance, find the form of an infinitely long cylinder so that the attraction may be a maximum at an external point.

Take the point for origin; pass a plane through it perpendicular to the generating lines of the cylinder. Let r be the radius vector of any point on this section, θ the angle made by r with the direction of the resultant attraction. The equation of the curve is included in $\dfrac{\cos\theta}{r^{n-1}} = C$.

When the law of attraction is the inverse square the required cylinder is right circular.

The Potential.

31. Let A_1, A_2, &c. be the positions of any number of fixed attracting particles; m_1, m_2, &c. their masses. The potential of these particles* at any proposed point P is defined to be

$$V = \frac{m_1}{r_1} + \frac{m_2}{r_2} + \&\text{c.} = \Sigma\,\frac{m}{r},$$

where r_1, r_2, &c. are their distances from P regarded as positive quantities.

This may be called the geometrical definition of the potential. Another definition founded on the principle of work will be given a little further on. In discussing the attractions of geometrical figures the former is the more convenient for use, but in many physical applications the latter will be found the more satisfactory.

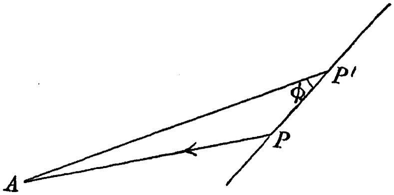
* The earliest use of the function now called the potential is due to Legendre in 1784, who refers to it when discussing the attraction of a solid of revolution. Legendre however expressly ascribes the introduction of the function to Laplace, and quotes from him the theorem connecting the components of attraction with the differential coefficients of the function. The name, Potential, was first used by Green in his *Essay on the application of Mathematical Analysis to the theories of Electricity and Magnetism*, published in 1828. Green also gave many of the theorems on this function now in continual use, but which have been since associated with the names of others who have discovered them a second time. Gauss also uses the name in Art. 3 of his memoir on *Forces acting inversely as the square of the distance*, Leipsic 1840, translated in the third volume of Taylor's *Scientific Memoirs*. The reader may also consult Todhunter's *History*, Art. 790, and Thomson and Tait's *Treatise on Natural Philosophy*, Art. 483.

We may notice that as the point P moves in space th potential is, by the definition, a continuous function of the positio of P. We must however except the case in which any one of th distances r_1, r_2, &c. vanishes or changes sign, for then the terı m/r ceases to represent the potential of the particle from which is measured.

32. If m be the mass of any one of the attracting particle: A its position, r its distance from a point P, the potential of m a P is $\frac{m}{r}$. Let P' be any point adjacent to P, and let $PP' = d$: The difference of the potentials of m at P and P' is then

$$\frac{d}{ds}\left(\frac{m}{r}\right) ds = -\frac{m}{r^2}\frac{dr}{ds} ds.$$

If ϕ be the angle $AP'P$, we have $\cos\phi = dr/ds$. The attractioı of m at P acts in the direction PA, and is equal to m/r^2; hence its resolved part in the direction PP' is $\frac{m}{r^2}\cos APP' = -\frac{m}{r^2}\frac{dr}{ds}$. Comparing this with the above result we see that if P, P' be two adjacent points, the excess of the potential at P' over that at P, divided by the distance PP is equal to the resolved attraction in the direction PP'.



This, being true for every particle of an attracting system, is necessarily true for the whole. We have therefore the following theorem. *If V, V' be the potentials of a system at two neighbouring points P, P', the attraction at P resolved in the direction PP' in which s is measured is the limit of* $\frac{V'-V}{PP'} = \frac{dV}{ds}$.

So long as the point P is situated outside the attracting mass the potentials V and V' are both finite and this proof is free from ambiguity. The case in which P lies within the attracting mass will be considered a little further on.

33. By taking the displacement PP' parallel to the axes of x, y, z in turn, we see that the components of the attraction in the *positive directions* of the axes are respectively

$$X = \frac{dV}{dx}, \quad Y = \frac{dV}{dy}, \quad Z = \frac{dV}{dz}.$$

In the same way the components of the attraction in polar

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32. If m be the mass of any one of the attracting particles, A its position, r its distance from a point P, the potential of m at P is $\dfrac{m}{r}$. Let P' be any point adjacent to P, and let $PP' = ds$. The difference of the potentials of m at P and P' is then

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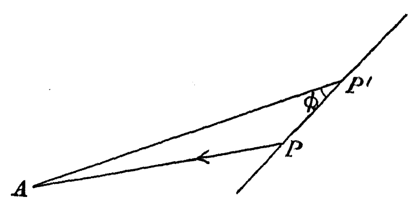

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So long as the point P is situated outside the attracting mass the potentials V and V' are both finite and this proof is free from ambiguity. The case in which P lies within the attracting mass will be considered a little further on.

33. By taking the displacement PP' parallel to the axes of x, y, z in turn, we see that the components of the attraction in the *positive directions* of the axes are respectively

$$X = \frac{dV}{dx}, \quad Y = \frac{dV}{dy}, \quad Z = \frac{dV}{dz}.$$

In the same way the components of the attraction in polar

coordinates may be expressed. Let r, θ, ϕ be the polar coordinates of any point P, let F, G, H be the components at P in the directions in which dr, $rd\theta$, $r\sin\theta d\phi$ are drawn, then

$$F=\frac{dV}{dr},\quad G=\frac{dV}{rd\theta},\quad H=\frac{dV}{r\sin\theta d\phi}.$$

34. It appears from this proposition that, when the potential V of a body fixed in space is given, its resolved attractions at any point P can be found by simply differentiating the potential with regard to the coordinates of that point. It follows that, if two different bodies have equal potentials throughout any space, they equally attract any particle placed in that space. Thus the attraction of a body is determined by the single function V instead of the three components X, Y, Z.

One chief reason for the use of the potential is that a body, so far as its quality of attraction is concerned, is analytically given by a single function without the necessity of stating either the form or the structure of the attracting body.

When the potential is used merely to find the forces, it is obvious that we may add an arbitrary constant to its value as defined in Art. 31. We then have $V=\Sigma\frac{m}{r}+C$, where C is the constant added. When the attracting bodies are finite, it is convenient to choose C so that V is zero at an infinite distance; this assumption makes $C=0$. When the attracting bodies extend to infinity, the potential, as defined in Art. 31, is sometimes found to contain an infinite constant. It may then be preferable to keep C arbitrary and to absorb into its value all constants not immediately required. There is a certain inconvenience in having different definitions of the potential for finite and infinite bodies, especially when we wish to proceed from one to the other as a limit. In stating the results therefore for the Newtonian law of force we shall adhere to the definition of Art. 31. In special cases such a constant may then be added as may most simplify the expression for V.

35. Ex. 1. If the law of force be the inverse nth power of the distance, show that the function $V=\frac{1}{n-1}\Sigma\frac{m}{r^{n-1}}$ is such that its differential coefficients with regard to x, y, z express the resolved forces at any point parallel to the coordinate axes. To this function we may of course add any constant.

Ex. 2. If the law of force be the inverse distance, then $V=-\Sigma m\log r$, together with any constant.

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Ex. 3. The law of force being the inverse square, find the values of the second differential coefficients of V at any point of space, and show that at the origin

$$\frac{d^2V}{dx^2}=\Sigma m\frac{3\cos^2\alpha-1}{r^3},\quad \frac{d^2V}{dxdy}=\Sigma m\frac{3\cos\alpha\cos\beta}{r^3},$$

where r is the distance of m from the origin, and α, β are the angles the distance r makes with the positive directions of the axes of x and y.

36. **Work and potential.** A definition of the potential may also be given founded on the principle of work. Referring to the figure of Art. 32, let a particle of unit mass travel along the elementary arc PP'. It has been already shown that the resolved attraction in the direction PP' is $\dfrac{dV}{ds}$. The work done by the attraction is therefore $\dfrac{dV}{ds}ds$. If the particle continue its journey along any curve, starting from some point P and arriving at some other point Q, the work done by the attraction is $\int dV=V_Q-V_P$, where V_P and V_Q are the potentials at P and Q. Thus the excess of the potential at Q over that at P is the work done by the attraction on a particle of unit mass as it travels by *any path* from P to Q.

If the attracting body is finite in all directions, the potential at a point P infinitely distant is zero. It follows that *the potential at any point Q is the work done by the attracting forces on a particle of unit mass as it travels from an infinite distance along any path to the point Q.*

In the same way the potential at Q is the work which must be done against the attraction by some external cause to move a unit particle from Q to an infinite distance.

The several particles of the attracting mass are supposed to remain fixed in space while the attracted particle makes its journey from P to Q.

37. **Level surfaces.** The locus of points at which the potential has any given value is called *a level surface.* It is also called *an equipotential surface.*

At any point of a level surface the resultant force acts along the normal to the surface.

To show this, let P_1 be a point on a level surface, and let P_2 be any neighbouring point also on the surface. If V_1, V_2 be the potentials at these points, the component force in the direction of any tangent P_1P_2 will be the limit of $\dfrac{V_2-V_1}{P_1P_2}$. This is zero since

$V_1 = V_2$. The resultant force must therefore act along the normal at P_1.

38. Let two neighbouring level surfaces be drawn at which the potentials are respectively $V_1 = c$ and $V_2 = c + \delta c$. *The normal attraction at any point P of either surface is inversely proportional to the length of the normal at that point intercepted between these level surfaces.*

To prove this, let the normal at any point P_1 on the first surface intersect the second surface in P_2. The normal force at P_1 is then ultimately
$$F = \frac{V_2 - V_1}{P_1P_2} = \frac{\delta c}{P_1P_2}.$$
It is therefore evident that F varies inversely as P_1P_2.

If a rigid surface were constructed having the form of a level surface and coincident with it, it is clear that a particle, placed at any point of the surface, would be pulled by the attracting body in a direction normal to the surface. The particle, if placed on the proper side, would therefore be in equilibrium. Level surfaces are therefore also called *surfaces of equilibrium*.

39. *A Line of force* is a curve such that the direction of the resultant force at any point is a tangent to the curve. It is evident that the whole system of level surfaces is cut orthogonally by the system of lines of force.

40. Ex. 1. A free particle placed at rest at any point of a line of force will move along the curve in such a direction that the potential increases.

Ex. 2. The level surfaces of a uniform straight rod are prolate spheroids having the extremities for foci. The lines of force are hyperbolas.

These results follow from the theorem that the attraction at P bisects the angle joining P to the extremities of the rod. Art. 13.

Ex. 3. Show that, if matter attracting according to the Newtonian law be arranged so that the direction of the resultant attraction at any external point P shall always pass through a fixed point O, the magnitude of the resultant attraction will be a function only of the distance OP, and will not depend on the angular coordinates of OP.

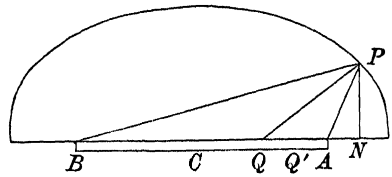
Ex. 4. The density of a straight line at any point P varies inversely as the square root of the product of the distances of P from the extremities A, B of the straight line, and the law of attraction is the inverse distance. Show that the level curves are ellipses having A and B for foci, and that the attraction at any point O varies inversely as the square root of the product $AO \,.\, BO$.

Of what body are the hyperbolas, having A, B for foci, the level curves?

[Coll. Ex.]

41. **Potentials of rods.** *To find the potential of a thin uniform rod AB at any point P.*

Let C be the middle point of the rod, $2l$ its length, p the length of the perpendicular PN. Let QQ' be an element of the length $CQ=x$, $CN=\xi$. Taking as before m for the mass per unit of length, the potential at P is $V=\int \frac{mdx}{PQ}$.



Since $PQ^2=(\xi-x)^2+p^2$, we find after substitution and integration between the limits $x=-l$ and $x=l$,

$$V=m\log\frac{\{(l-\xi)^2+p^2\}^{\frac{1}{2}}+l-\xi}{\{(l+\xi)^2+p^2\}^{\frac{1}{2}}-l-\xi}=m\log\frac{r-\xi+l}{r'-\xi-l},$$

where $r=AP$ and $r'=BP$.

Describe an ellipse whose foci are A and B and which passes through the point P. We may obtain a simpler expression for V by using some of the properties of this ellipse. Substituting $r=a-e\xi$, $r'=a+e\xi$ and $l=ae$ and cancelling out the common factor $a-\xi$, we find $V=m\log\frac{1+e}{1-e}$. If we now multiply numerator and denominator by $2a$ and substitute $r+r'$ for $2a$ and l for ae, we have $V=m\log\frac{r+r'+2l}{r+r'-2l}$. It appears from either of the two last results that the potential is constant at all points situated on the prolate spheroid whose foci are the extremities A, B of the rod.

42. When the rod is infinite in length the potential is easily deduced from the attraction already found in Art. 14. Since the magnitude of the attraction is $\frac{2m}{p}$ and its direction is PN, it is evident that the potential must be $V=C-2m\log p$, where C is a constant.

We may also deduce this result from the expression for the potential of a finite rod. Suppose the point P to be situated in the straight line drawn through C perpendicular to the rod. Then $\xi=0$ and $r'=(l^2+p^2)^{\frac{1}{2}}=l+\frac{1}{2}\frac{p^2}{l}$ We then have

$$V=m\log\frac{r+l}{r'-l}=2m\log 2l-2m\log p.$$

We thus see that *the constant C is really infinite and equal to* $2m\log 2l$ when we adhere to the definition of Art. 31.

43. Ex. 1. A thin straight rod AB infinite in the direction of B has one extremity at the point A; show that the potential at a point P is given by

$$V = C - m \log (AP + AN),$$

where AN is measured positively in the direction BA, and C is an infinite constant.

Ex. 2. Show that the potential of a thin rod AB at any point P is

$$V = m \log (\cot \tfrac{1}{2} PAB \,.\, \cot \tfrac{1}{2} PBA).$$

Ex. 3. A thin uniform rod AB is attracted by a body of any form: show that the component of the attraction along the length BA of the rod is $m(V_A - V_B)$, where V_A and V_B are the potentials of the body at A and B, and m is the mass of the rod per unit of length.

By Art. 11 this theorem is true when the rod is attracted by a single particle; it is therefore true by summation when attracted by any body.

Ex. 4. A uniform thin chain AB is enclosed in a smooth curvilinear tube which it just fits, and is attracted by a body of any form. Show that the force urging the chain to move in the tube is $m(V_A - V_B)$. Hence show that the position of equilibrium may be found by equating the potentials of the body at the extremities of the chain.

That the force depends only on the positions of the extremities of the chain and not on its length or form may also be shown by another kind of reasoning. Let the chain be completed into a circuit by uniting two chains in different tubes at their extremities. If the forces were not equal the chain would begin to move round the circuit and thus a perpetual motion would be caused by the mere presence of an attracting body.

Ex. 5. Prove that the surface, over which the potential of an attracting rod of length $2c$ and density unity is equal to a given quantity V, is represented by the equation $x^2 (e^{\frac{1}{2}V} - e^{-\frac{1}{2}V})^2 + \frac{1}{4}(y^2 + z^2)(e^V - e^{-V})^2 = c^2 (e^{\frac{1}{2}V} + e^{-\frac{1}{2}V})^2$. [Math. T.]

44. Ex. 1. Two equal particles, each of mass m, are placed at two fixed points A, B whose distance apart is $2a$. The particle at A repels and that at B attracts a particle placed at any point P. If r be the distance of P from the middle point C of AB and θ be the angle PCA, prove that the potential, due to both particles, at P is equal to $\dfrac{-2am\cos\theta}{r^2}$ nearly, where r is very great compared with a.

To prove this we notice that

$$V = \frac{-m}{AP} + \frac{m}{BP} = \frac{-m}{r - a\cos\theta} + \frac{m}{r + a\cos\theta} = \frac{-2am\cos\theta}{r^2}.$$

A system of two equal particles, one attracting and the other repelling, is called a magnet. The points A, B are the poles, and $-2am$ is called the magnetic moment, the negative sign being given, because it is found more convenient to make repulsion instead of attraction the standard case. Representing the magnetic moment by M, *the potential of a small magnet is* $\dfrac{M\cos\theta}{r^2}$.

Ex. 2. The north pole N of a magnet attracts a particle, and the south pole S repels it according to the law of the inverse square, the absolute forces of the two poles being equal: prove that the lines of force are symmetrical curves, concave to the magnet and passing through its poles. If P be the middle point of one of the lines of force NPS, prove that the curvature at P is three-halves that of the circle

NPS, and that the curvatures at N and S are zero. If NPS be an equilateral triangle, prove that the line of force meets the magnet at right angles.

[Math. Tripos, 1871.]

, 236

Ex. 3. Three small magnets are placed with their centres at the angular points of an equilateral triangle ABC, and being free to move about those centres rest in the following positions. The magnet at A is parallel to BC, whilst those at B and C are at right angles to AB, AC respectively. Show that the magnetic moments are in the ratios $\sqrt{3} : 4 : 4$. [Math. Tripos, 1880.]

J. Thom. ii. 128.

Ex. 4. Two magnetic particles, of moments m and m', are fixed at two corners of an equilateral triangle with their axes bisecting the angles. A third magnetic particle is free to move at the other angular point. Show that its axis m[illegible] the bisector of the third angle an angle $\tan^{-1} \dfrac{\sqrt{3}}{7} \dfrac{m \sim m'}{m+m'}$. [Math. Tripos, 1882.]

45. Ex. A number of infinite straight attracting rods are arranged at equal distances on the surface of a cylinder of radius a. If n be the number of rods, m the mass of each per unit of length, prove that their potential at any point P is given by

$$V = C - m \log (r^{2n} - 2a^n r^n \cos n\theta + a^{2n}),$$

where r is the distance of P from the axis of the cylinder and θ the angle r makes with a plane through the axis and one of the attracting rods.

By making n infinite while the whole mass is given, show that the potential of a uniform thin cylindrical shell at the point P is $C - 4\pi a M \log a$ or $C - 4\pi a M \log r$ according as P is inside or outside the cylinder, the mass per unit of area being M.

These expressions follow from Art. 42 by using De Moivre's property of the circle.

These results are of considerable interest because they help us to understand how the potential of a thin cylindrical shell is a discontinuous function of the coordinates, being constant at all points within the cylinder and depending on the logarithm of the distance from the axis at points outside. Supposing the number of rods to be very great but not infinite, the potential at any point P is represented by a continuous function of the coordinates of P, i.e., as P travels from the interior to the exterior through the interstices between the rods the potential is always the same function of the coordinates. When P is inside the cylinder, r/a is less than unity, and by expanding the logarithm in powers of r/a we see that

$$V = C - 2mn \log a + 2m \left(\frac{r}{a}\right)^n \cos n\theta + \&c.$$

It follows that when n is large the potential is sensibly constant throughout the interior except in the immediate neighbourhood of the surface of the cylinder on which the rods lie. When P is outside, a/r is less than unity and by expanding the logarithm in powers of a/r we find $V = C - 2mn \log r + 2m \left(\dfrac{a}{r}\right)^n \cos n\theta + \&c.$ It appears that, except in the immediate neighbourhood of the surface of the cylinder, the potential when n is large does not sensibly differ from $C - 2mn \log r$ at any point outside.

As n increases the small space within which the potential differs from the first term of these series gets continually less, and in the limit is zero, so that we may say that the potential is constant throughout the interior of the cylinder and, except for C, varies as the logarithm of the distance throughout external space.

46. Ex. 1. The space within a closed surface S is filled with homogeneous matter of density unity: prove that the potential V at any point P is given by $V=\frac{1}{2}\int \cos\phi d\sigma$, where $d\sigma$ is the area of an element of the surface S at any point Q, and ϕ is the angle the normal at Q drawn inwards makes with the distance QP.

[Smith's Prize, 1871.]

With P as vertex describe a cone whose base is $d\sigma$, and let $d\omega$ be the solid angle. The potential of an element of the cone distant R from the vertex is $R^2 d\omega dR/R$. Integrating this from $R=0$ to $R=r$, where $PQ=r$, the potential of the elementary cone is $\frac{1}{2}r^2 d\omega$. This is easily seen to be $\frac{1}{2}\cos\phi d\sigma$.

Ex. 2. Show that the volume of the solid enclosed by the surface S is [illegible], where $r=PQ$. [Gauss' Theorem.]

Ex. 3. Show that $\int \frac{\cos\phi}{r^2} d\sigma$ is equal to 4π, 2π, or 0 according as the point P is inside, on the surface S, or outside. [Gauss' Theorem.]

47. **Potentials of discs and cylinders.** *To find the potential of a circular disc at any point P situated in its axis.*

Referring to the figure of Art. 21, the potential at P of the annulus QQ' is $2\pi mx dx/PQ$, where x and $x+dx$ are the radii of the annulus and m the mass of the disc per unit of area. If p be the perpendicular from P on the disc and r the distance PQ, we have $r^2 = x^2 + p^2$ and $rdr = xdx$. Substituting, we find that the potential V of the disc is $V = 2\pi m \int dr = 2\pi m (r_1 - p)$, where r_1 is the distance from P of any point on the perimeter.

If a be the radius of the disc, we may also write this in the form

$$V = 2\pi m \{\sqrt{a^2+p^2} - p\}.$$

Ex. 1. Show that the potential of a thin disc of infinite area at a point distant p is $A - 2\pi mp$, where A is an infinite constant.

Ex. 2. Show that the potential of a circular cylinder of density ρ, radius a, and small thickness h at an external point P on the axis close to the cylinder is $2\pi\rho h(a-p)$, where p is the mean of the distances of P from the two plane faces of the cylinder.

Ex. 3. Prove that the potential of a circular disc of radius a and unit density at a point in its plane distant R from the centre is

$$\int_0^{2\pi} \frac{(a^2 - aR\cos\theta)\, d\theta}{\sqrt{(a^2 - 2aR\cos\theta + R^2)}}.$$ [Math. Tripos, 1884.]

Ex. 4. The particles of a thin uniform circular ring attract a particle P situated in its own plane according to the law of the inverse cube. Show that the resultant attraction on P is $\pm \frac{mr}{(r^2-a^2)^2}$, where m is the mass of the ring, a its radius, and r the distance of its centre from P. Which sign should be given to this expression? [Townsend's Problem.]

Ex. 5. At the focus of a thin shell in the form of a paraboloid of revolution the potential of any portion bounded by planes perpendicular to the axis varies as $R_1 - r_1$ if the density be constant, and as $R_2 - r_2$ if the density vary as the focal

distance; where R_1, R_2 are the principal radii of curvature at one edge, r_1, r_2 at the other edge.

48. Ex. 1. An indefinitely thin layer of attracting matter is placed on an infinitely long circular cylinder of radius a, so that the density ρ is uniform along any generating line, but varies from one generating line to another. Let the axis of the cylinder be the axis of reference, and let the cylindrical coordinates of any point be (r, ϕ, z). If $\rho = A\cos n\phi + B\sin n\phi$, where n is any integer except zero, prove that the potential at a point whose coordinates are (r', ϕ', z') is equal to

$$\frac{2\pi a}{n}(A\cos n\phi' + B\sin n\phi')\left(\frac{a}{r'}\right)^{\mp n},$$

together with a constant, the upper or lower sign being taken according as the attracted point is inside or outside the cylinder. Find also the potential when ρ is any function of ϕ whatever.

Let P be the point at which the potential is required, and let PO be a perpendicular on the axis. Through P draw a plane cutting the cylinder in a circle; let A be a fixed point on the circle, QQ' an element of the circle, then $AOQ = \phi$; also $AOP = \phi'$, $OP = r'$. Let $\psi = \phi - \phi'$, $R = PQ$.

The matter placed on the generators which pass through the elementary arc QQ' may be regarded as an attracting rod whose potential at P is $-2m\log R + C$, where m is the mass per unit of length and C is a constant. The potential of the whole cylindrical stratum is therefore

$$\begin{aligned} V &= -\int a\,d\psi\,.\,\rho\log R^2 + C \\ &= -\int a\,d\psi\,.\,\rho\log(a^2 - 2ar\cos\psi + r^2) + C, \end{aligned}$$

where the limits are 0 and 2π and C is a constant.

Now by writing for $2\cos\psi$ its exponential value we easily find

$$\log(1 - 2h\cos\psi + h^2) = -2\,(h\cos\psi + \tfrac{1}{2}h^2\cos 2\psi + \tfrac{1}{3}h^3\cos 3\psi + \&c.),$$

and this series is convergent when h is less than unity.

To obtain a convergent series we must expand the logarithm in the integral for V in powers of r/a or a/r according as P is inside or outside the cylinder. We therefore write the potential in the forms

$$V = -\int a\,d\psi\,.\,\rho\log\left\{1 - 2\frac{r}{a}\cos\psi + \left(\frac{r}{a}\right)^2\right\} - \int a\,d\psi\,.\,\rho\log a^2 + C,$$

$$V = -\int a\,d\psi\,.\,\rho\log\left\{1 - 2\frac{a}{r}\cos\psi + \left(\frac{a}{r}\right)^2\right\} - \int a\,d\psi\,.\,\rho\log r^2 + C,$$

according as P is inside or outside the cylinder.

Suppose first that $\rho = L\cos n\psi$. Then, remembering that $\int\cos n\psi\cos m\psi\,d\psi = 0$ or π according as m and n are unequal or equal when the limits are 0 and 2π, we easily find $V = L\dfrac{2\pi a}{n}\left(\dfrac{r}{a}\right)^n + C$ or $L\dfrac{2\pi a}{n}\left(\dfrac{a}{r}\right)^n + C$, according as P is inside or outside.

Next suppose $\rho = L\sin n\psi$, then since $\int\cos n\psi\sin m\psi\,d\psi = 0$ when the limits are 0 and 2π, we find by the same reasoning as before that the potential at P is constant whether P is inside or outside.

Lastly let ρ have its given value, viz.

$$\rho = A\cos n(\psi + \phi') + B\sin n(\psi + \phi') = L\cos n\psi + M\sin n\psi,$$

where $\qquad L = A\cos n\phi' + B\sin n\phi'$ and $M = -A\sin n\phi' + B\cos n\phi'$.

We then find by using the above results that

$$V = \frac{2\pi a}{n}L\left(\frac{r}{a}\right)^n + C \text{ or } \frac{2\pi a}{n}L\left(\frac{a}{r}\right)^n + C,$$

according as P is inside or outside the cylinder.

If ρ be any function of ϕ, then for all values of ϕ between 0 and 2π we may expand ϕ by Fourier's theorem in a series of the form $\rho = \Sigma \{A_n \cos n\phi + B_n \sin n\phi\}$.

The potentials due to each term may be separately found and the results added together.

Ex. 2. A uniform thin stratum of attracting matter is placed on an infinite right circular cylinder. Show (1) that the potential at any internal point is the same as that at the axis, (2) that the potential at any external point is the same as if the whole mass of the stratum were uniformly distributed over the axis.

Ex. 3. The density of a thin stratum on a right circular cylinder of radius a is proportional to the distance from a plane through the axis and its greatest value is D. Prove that the potential at any point P is $2\pi a^2 D \frac{\xi}{r^2}$ or $2\pi D\xi$ according as P is outside or inside, where ξ and r are the distances of P from the given plane and from the axis respectively.

49. **Systems of particles.** If a particle of mass m_1' travel from a position at which the potential is zero along any path to any assigned position B_1, it is clear from what precedes that the work done by the attracting forces is $V_1 m_1'$, where V_1 is the potential at B_1. If a second particle m_2' travel from a position of zero potential to the position B_2, it is clear that the additional work is $V_2 m_2'$, where V_2 is the potential at B_2 of the same attracting forces.

Generalizing this, let there be two systems of particles; let the masses of the first be m_1, m_2, &c., and let these be situated at the points A_1, A_2, &c. Let the masses of the second be m_1', m_2', &c. and let these be situated at the points B_1, B_2, &c. Let V_1, V_2, &c. be the potentials of the first system at B_1, B_2, &c.; V_1', V_2', &c. the potentials of the second system at A_1, A_2, &c. Let us also suppose that each particle of either system acts on all the particles of the other but does not attract any particle of its own system. The work done by the attracting forces in moving the particles of the second system from positions of zero potential to their assigned positions is

$$W' = V_1 m_1' + V_2 m_2' + \dots$$

In the same way the work of bringing the particles of the first system from positions of zero potential to the positions A_1, A_2, &c. under the influence of the attracting forces of the second system is

$$W = V_1' m_1 + V_2' m_2 + \dots$$

If r_{12} be the distance between the particles m_1, m_2', and r_{21} that between the particles m_2, m_1', and so on, the values of the potentials V_1, V_1' are $\quad V_1 = \frac{m_1}{r_{11}} + \frac{m_2}{r_{21}} + \text{\&c.},$

$$V_1' = \frac{m_1'}{r_{11}} + \frac{m_2'}{r_{12}} + \&c.$$

Substituting, we find that each of the expressions W, W' is equal to $$\frac{m_1 m_1'}{r_{11}} + \frac{m_1 m_2'}{r_{12}} + \frac{m_2 m_1'}{r_{21}} + \ldots = \Sigma \frac{mm'}{r}.$$

This symmetrical expression is sometimes called *the mutual potential* and sometimes *the mutual work* of the two systems.

The work required to move either system from one given position to another under the influence of the attractions of the other system is the difference of their mutual potentials in the two positions. *If both systems are moved, each from one given position to another, under the influence of their mutual attractions,* it easily follows, by moving them one at a time, that *the work done is the excess of their mutual potential in their final positions over that in their initial positions.*

50. If the particles are elements of a solid body the argument is still the same. Let dv' be an element of the volume of any finite mass M', ρ' its density, V the potential of any fixed system of attracting bodies; the work of collecting together the mass M' is $\int V\rho' dv'$.

This formula may be put into the form of a rule. *To find the mutual potential of two attracting masses in assigned positions, we multiply the mass of each element of one body by the potential of the other at that element, and then integrate the result throughout the volume of the first body.*

51. The particles of a system mutually attract each other and are in assigned positions. Supposing them to have been originally at distances so far apart that their mutual attractions were zero, it is required to find the work done by their attractions as they are collected together and brought each into its assigned position.

Let us begin by bringing the first particle m_1 into its assigned position A_1; as there are no other particles of the system sufficiently near to exert attraction on this particle the work required is zero. If we now conduct the second particle m_2 into the position A_2, the work done by the attraction of m_1 is $\frac{m_1 m_2}{r_{12}}$, where r_{12} is the distance A_1A_2. In bringing the third particle m_3 into the position A_3 the work done by the attractions of m_1 and m_2 is $\frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}$ and so on.

The whole work done by the attractions when every particle of the system is brought into its assigned position is therefore

$$W = \Sigma \frac{m_1 m_2}{r_{12}} \text{ ....(1).}$$

Let V_1, V_2, &c. be the potentials at A_1, A_2, &c. of the whole system after every particle has been brought into its assigned position, then

$$V_1 = \frac{m_2}{r_{12}} + \frac{m_3}{r_{13}} + \text{\&c.,} \quad V_2 = \frac{m_1}{r_{12}} + \frac{m_3}{r_{23}} + \text{\&c.}$$

We may then transform the expression (1) into

$$W = \tfrac{1}{2}(V_1 m_1 + V_2 m_2 + \dots) = \tfrac{1}{2}\Sigma Vm \text{........... (2).}$$

The term $m_1 m_2/r_{12}$ occurs twice in the expression (2), viz. once in $V_1 m_1$ and once in $V_2 m_2$, but it only occurs once in the expression (1). We have therefore to introduce the factor $\frac{1}{2}$ in the expression (2) to make the result agree with the value of W given in (1).

The rule to find the mutual potential of a single system of attracting particles is therefore slightly different from that given in Art. 49 to find the mutual potential of two different systems.

To find the mutual potential of a system of attracting particles brought from infinite distances to any assigned positions, we multiply the mass of each element by the potential at that element, integrate throughout the volume and halve the result.

This rule, when the final sign is reversed, also gives the work of bringing the particles from any assigned positions to infinite distances. To find the work of bringing the particles from one assigned arrangement to another, we add together the work of bringing them from the first arrangement to infinite distances and the work of bringing them from infinite distances to the second arrangement. If the system be moved, like a rigid body, from one place to another so that the relative positions of the particles in the two places are the same, it is clear that no work is done by the mutual attractions of the particles.

In this investigation we have treated the elementary masses as if their linear dimensions were infinitely small compared with their distances apart. It might therefore be supposed that the argument fails for two elements of a continuous body which finally become contiguous. We may however notice that it is only infinitely small portions of adjacent elements which can be in contact, and we shall now prove that, when the density ρ is finite, the mutual potential of these portions tends to zero as their distance apart decreases. If r be the very small distance between the nearest points of two elements, the only portions of each, whose distance apart is of the order r, have masses of the order ρr^3. Dividing these

portions into smaller elements having their linear dimensions infinitely smaller than r we see that the mutual potential of these two portions is of the order $\rho^2 r^5$ and therefore vanishes in the limit when the elements become contiguous. Since the portions not in contact can be divided into elements so small that their linear dimensions are infinitely smaller than their distance apart, the rule proved above will apply for the rest of the matter, and therefore for the whole of the bodies.

52. It appears from the definition of potential that its dimensions are not the same as those of work. The potential of a particle whose mass is m at a point P distant r is $\frac{m}{r}$. If a particle of mass m' is situated at the point P, the mutual potential or work of these two particles is $\frac{mm'}{r}$. The dimensions of the first are therefore mass divided by distance, those of the second mass squared divided by distance.

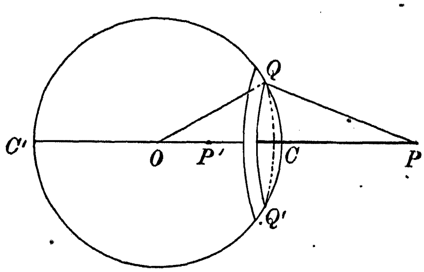
Spherical Surface.

53. *To find the potential of a thin uniform spherical shell at any point.*

Let O be the centre of the shell, a the radius of either bounding surface, m the mass per unit of area. Let P be the point at which the potential is required, $OP = R$.

Taking on the surface of the shell an annulus QQ' whose axis is OP, let the angle

$POQ = \theta$, and $QP = u$.



Since the mass of the annulus is $m \,.\, ad\theta \,.\, 2\pi a \sin\theta$ by Pappus' theorem (Vol. I. Art. 413), the potential at P of the whole shell is

$$V = \int \frac{2\pi m a^2 \sin\theta \, d\theta}{u}.$$

Since $u^2 = R^2 + a^2 - 2aR\cos\theta$, we have $udu = aR\sin\theta d\theta$.

Substituting, we find $V = \frac{2\pi ma}{R} \int du$.

If the point P is external to the surface as shown in the figure, the limits of u are $u = PC$ to $u = PC'$, i.e. $u = R - a$ to $R + a$. In this case $V = \frac{4\pi ma^2}{R}$.

If the point P is inside the shell as at P', the limits of u are $u = P'C$ to $u = P'C'$, i.e. $u = a - R$ to $a + R$. In this case

$$V = \frac{4\pi m a^2}{a}.$$

If M be the whole mass of the shell, $M = 4\pi m a^2$, and these expressions take the form $V = \frac{M}{R}$ or $V = \frac{M}{a}$ according as the attracted point P lies outside or inside the shell.

When the point P is at the centre, u is constant and cannot be properly taken as the independent variable. But since every element of the attracting mass is equally distant from P, it is evident that the potential at the centre is equal to the mass divided by the radius, and this agrees with the above result.

54. Since the potential is the same at all points within the spherical shell, it follows that its differential coefficient with regard to each of the coordinates is zero. Thus *the attraction of a thin uniform spherical shell at an internal point is zero.*

Since a thick shell bounded by concentric spheres may be regarded as composed of a sufficient number of thin shells, it follows that *the attraction of a thick shell bounded by concentric spheres at an internal point is zero.*

This theorem is also true for *a heterogeneous thick shell provided the strata of equal density are concentric spheres.* For in this case each of the thin shells into which it is analysed is homogeneous.

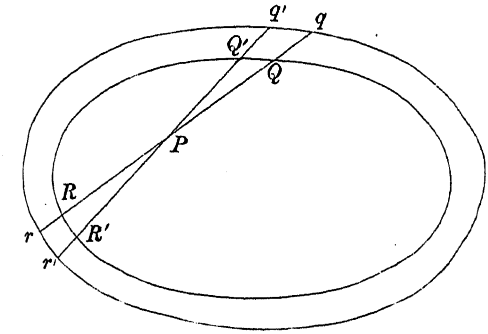
55. Since the potential at an external point of a uniform thin shell is M/R, we see that the force at an external point P resolved in the direction OP is equal to $-M/R^2$. *The attraction therefore acts in the direction from P towards the centre, and is the same as if the whole mass were collected at its centre.*

As before, since a thick shell may be analysed into elementary thin shells, it follows that *the attraction of a thick shell bounded by concentric spheres or of a solid sphere at any external point is the same as if the whole mass were collected into its centre. Also this is true for heterogeneous shells provided the strata of equal density are concentric spheres.*

These theorems on the attraction of a spherical shell as well as that of a spheroid at an internal point are due to Newton.

56. That the attraction of a thin uniform shell bounded by concentric spheres at an internal point P is zero may be shown by an elementary geometrical argument which applies also to the case of some ellipsoidal shells.

With P as vertex describe an elementary cone cuttin surfaces of the shell in $QQ'qq'$, $RR'rr'$ respectively. Let P $Qq = dr$; $PR = r'$, $Rr = dr'$. If $d\omega$ be the solid angle o elementary cone, the volumes of the elementary solids at Q a will be respectively $r^2 d\omega\, dr$ and $r'^2 d\omega\, dr'$. Their attractions are therefore $\rho d\omega\, dr$ and $\rho d\omega\, dr'$, where ρ is the density. The tractions will balance each other whenever the form of the sl such that the intercepted parts Qq, Rr of the chord qQR equal. This being true for all chords through P, the attracti



every element is balanced by that of the opposite element an resultant attraction on P is zero.

When the shell is bounded by concentric spheres thes tercepted parts are evidently equal. The resultant attractic any internal point is therefore zero.

When the shell is bounded by similar and similarly siti concentric ellipsoids the same is also true. To prove thi notice that, since the chords parallel to QR have in the ellipsoids a common diametral plane, the chords QR and qr have the same middle point. It follows that the intercepted Qq and Rr are equal.

Since a thick shell may be analysed into elementary shells, it follows that *the attraction of any homogeneous bounded by similar and similarly situated concentric ellip at any internal point is zero.*

57. If P is on the outside of a thin ellipsoidal shell, bou by similar concentric ellipsoids, we may show by similar reasc that *the enveloping cone whose vertex is P divides the surface two portions whose attractions at P are the same in direction n magnitude.*

It follows that, when P is indefinitely close to the outer margin the shell, the infinitely small portion on the nearer side of the lar plane exerts the same attraction at P as all the rest of the ell. If the thin shell is spherical, the resultant attraction is own to be the same as if the whole mass were collected at its ntre. Putting m for the mass per unit of area, the attraction P of each of the portions on the two sides of the polar plane is m.

58. We may apply these results to the solid bounded by two centric similar and similarly situated hyperboloids. If one et attract and the other repel, the attraction on P is zero, vided both sheets are on the same side of P.

Also a paraboloidal shell bounded by two equal paraboloids ing their axes coincident but their vertices separate exerts no action at an internal point.

59. If the thin shell is ellipsoidal and P is very close to the er margin, the distance of P from the polar plane is infinitely ller than the linear dimensions of the curve of contact. The action at P of the portion on the nearer side of the polar plane erefore the same as that of an infinite plate of the same thick-, see Art. 22. The attraction at P of each of the portions on two sides of the polar plane is therefore $2\pi m$, where m is the s of the shell in the neighbourhood of P per unit of area. The action of the whole shell at a point P, just outside the shell, is efore twice that of an infinite plate of the same thickness as of the shell at P, i.e. the attraction is $4\pi m$. It also follows the direction of the attraction is the same as that of the ite plate and is normal to the shell. This line of argument be more fully considered further on.

. Ex. 1. A thin stratum of matter is placed on a complete right cone so he surface density at any point is inversely proportional to the distance from ertex, the matter on one side of the vertex attracting, that on the other ing. Show that the stratum exerts no attraction at a point having both on the same side.

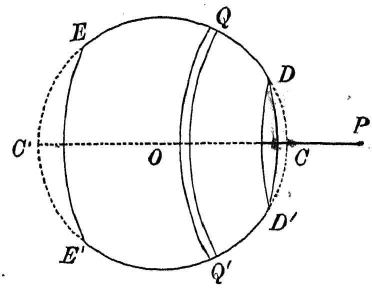
. 2. If matter attracting according to the law of gravitation be uniformly uted upon the circumference of a circle, show that the chord of contact gents drawn to the circle from any external point divides the circle into rcs, such that the potentials at the point due to each arc are the same. [Math. Tripos.]

1. **Potential of an annulus.** We may use the method of

Art. 53 to find the potential of an annulus of a thin uniform spherical shell at a point P on its axis.

Let $DD'EE'$ be the portion of the spherical shell whose potential at P is required. Let $PD=u_1$, $PE=u_2$; $OP=R$.

The potential of an elementary annulus QQ' being the same as before, the potential V of the whole annulus is



$$V=\frac{2\pi ma}{R}\int du=\frac{2\pi ma}{R}(u_2-u_1),$$

since in our case the limits of integration are $u=PD$ and $u=PE$. In the same way the mass M of the annulus is

$$M=\frac{2\pi ma}{R}\int u\,du=\frac{\pi ma}{R}(u_2^2-u_1^2).$$

We have therefore for the potential of the whole given annulus

$$V=\frac{M}{\frac{1}{2}(u_1+u_2)}.$$

62. If we suppose the annulus to form a complete sphere except for two small holes DD', EE', we have an expression for the potential which applies equally to points inside and outside the shell, provided they lie on the axis. Let y be the radius of either hole. When P is inside the shell the sum of the distances u_1 and u_2 differs from the diameter only by small quantities of the order y^2 and the potential is therefore sensibly constant. When P passes through the hole DD' the distance u_1 has a minimum value equal to y and then begins to increase without vanishing or changing sign. When P is outside the shell the sum of the distances u_1 and u_2 differs from twice the distance of P from the centre by quantities of the order y^2, so that the potential sensibly follows the law of the inverse distance.

In the limit, when the holes are closed, the potential of a thin spherical shell at a point P is given by $V=\frac{M}{\frac{1}{2}(u_1+u_2)}$, where u_1 and u_2 are the distances of P from the extremities of the diameter on which P lies. This expression will apply to points both inside and outside, provided we assume that the distances u_1 and u_2 are taken positively for all positions of P. When P passes through the shell from one side to the other this assumption makes the expression discontinuous in form.

The case of the annulus is similar to that of the cylinder of rods considered in Art. 45. The collection of rods have a continuous potential both inside and outside the cylinder, and this becomes discontinuous in form when the rods form a complete cylinder.

63. Ex. 1. The attraction of an annulus of a thin spherical shell at any point P on the axis is $\frac{2\pi ma}{r^2}(\chi\pm\chi')$, where r is the distance of P from the centre and χ, χ' are the halves of two chords of the sphere, both of which pass through P and one

through each rim of the annulus. The negative or positive sign is to be taken according as the rims are on the same or opposite sides of the polar plane of P.

Ex. 2. From a spherical shell of small thickness $2t$ and uniform density a segment is cut off by a plane. Prove that the potentials of the segment at two points on the axis, one just inside and the other just outside, are the same up to the first power of t.

Ex. 3. A thin spherical shell of radius a attracts an internal particle P at a distance R from the centre. If the shell be divided into two parts by a plane through P perpendicular to the radius the resultant attraction of each part at P is $\frac{2\pi ma}{R^2}\{a-(a^2-R^2)^{\frac{1}{2}}\}$ where m is the surface density. [Todhunter's *History*, 1615.]

Ex. 4. If the law of attraction be the inverse nth power of the distance, show that the potential of a thin spherical annulus at a point on its axis is

$$V=\frac{2M}{(n-1)(n-3)}\,\frac{u_2^{n-3}-u_1^{n-3}}{u_2^2-u_1^2}\,\frac{1}{(u_1u_2)^{n-3}},$$

following the notation of Art. 53.

If the law be the inverse cube, the potential is $V=M\,\dfrac{\log u_2-\log u_1}{u_2^2-u_1^2}$.

64. **A Solid Sphere.** *To find the attraction of a solid uniform sphere at an internal point P.*

Describe a sphere concentric with the given surface to pass through P. The attraction at P of the matter between this sphere and the given surface is zero; Art. 54. The attraction at P of the matter within this sphere is the same as if it were collected at the centre, Art. 55. If R be the distance of P from the centre O, the attraction is $\frac{4}{3}\pi\rho R^3/R^2$, where ρ is the density. It follows that *the attraction of a solid homogeneous sphere at an internal point distant R from the centre is $\frac{4}{3}\pi\rho R$.*

If (x, y, z) be the coordinates of P referred to the centre as origin, X, Y, Z the components of attraction, we have also

$$X=-\tfrac{4}{3}\pi\rho x,\qquad Y=-\tfrac{4}{3}\pi\rho y,\qquad Z=-\tfrac{4}{3}\pi\rho z.$$

These are obtained by resolving the resultant attraction, viz. $\frac{4}{3}\pi\rho r$, parallel to the axes.

65. We may apply the same method *to find the potential of a solid sphere at an internal point P.*

If x and $x+dx$ are the radii of an elementary shell taken within the sphere passing through P, its potential at P is $4\pi\rho x^2dx/R$, Art. 53. In the same way, if y and $y+dy$ are the radii of an elementary shell outside the same sphere, its potential at P is $4\pi\rho y^2dy/y$, Art. 53.

The potential at P of the whole sphere is therefore

$$V=\int_0^R\frac{4\pi\rho x^2dx}{R}+\int_R^a\frac{4\pi\rho y^2dy}{y}.$$

If the density ρ of the sphere is uniform this integral becomes

$$V = \frac{2\pi\rho}{3}(3a^2 - R^2).$$

If the density is any function of the distance from the centre the integration can be effected when the function is given.

66. Ex. 1. A portion of a homogeneous spherical shell is cut off by a cone whose vertex is at the centre and whose solid angle is $d\omega$. Show that the accelerating attraction of the rest of the shell on this portion is

$$\pi\rho(b-a)\frac{b^2+2ab+3a^2}{b^2+ab+a^2},$$

where a and b are the internal and external radii of the shell. Hence show that when the shell is indefinitely thin the accelerating attraction is *half* that just outside.

Since the resultant attraction of a body on itself is zero, the attraction of the rest of the shell is the same as that of the whole shell. The attraction on the portion included is $\int \frac{4\pi}{3}\rho^2 . \frac{r^3-a^3}{r^2} r^2 dr d\omega$; dividing this by the mass attracted, viz. $\frac{1}{3}\rho(r^3-a^3)\,d\omega$, we have the result above given.

Ex. 2. Prove that the pressure per unit of length on any normal section of a spherical shell of mass M and radius a due to the mutual gravitation of the particles tends to the limit $M^2/16\pi a^3$, as the thickness of the shell is indefinitely diminished. [Math. Tripos.]

Ex. 3. A solid homogeneous sphere is divided by a plane through its centre into two hemispheres. These being placed with their plane faces coincident, show that the force required to pull them apart is $\frac{3}{16}M^2/a^2$, where M is the mass of the sphere and a its radius.

Ex. 4. If the density of a solid sphere vary as the nth power of the distance from the centre, show that the potential at an internal point is

$$V = \frac{4\pi\rho}{(n+2)(n+3)}\left\{(n+3)a^2 - \frac{R^{n+2}}{a^n}\right\},$$

where ρ is the surface density and $n+2$ is positive.

Ex. 5. A homogeneous sphere is divided into two parts by a plane QNR bisecting OP at right angles, P being any point within the sphere and O the centre. If a be the radius of the sphere and $c = OP$, prove that the attraction at P of the larger part of the sphere cut off by the plane $QNR = \frac{3a-c}{4c} \times$ attraction at P of the whole sphere.

Ex. 6. A solid sphere of radius a has a hole pierced through it in the form of a right circular cylinder of radius b, the axis of the cylinder being a diameter of the sphere. Show that the potential V of the remaining solid portion at any point P of the axis is given by

$$\frac{V}{\pi\rho} = \frac{2}{3}\frac{PE^3 - PD^3}{OP} - (PN.PE - PM.PD) - b^2 \log\frac{PN+PE}{PM+PD},$$

where D, E are any points on the two circular rims of the cylinder; M, N the centres of the rims and O the centre of the sphere.

Show that when $b=0$, the right hand side reduces to $\frac{4}{3}a^3/R$ or $\frac{2}{3}(3a^2-R^2)$, where $R = OP$, according as P is without or within the sphere.

Ex. 7. If I be an external point and C the centre of a sphere, prove that the sphere on IC as diameter, the sphere with centre I and radius IC or the polar plane of I will divide the sphere into two parts exerting equal attractions at I, according as the law of attraction is the inverse square, the inverse cube, or the inverse fourth power of the distance. [St John's Coll., 1885.]

If the law be the inverse nth power, and a radius vector from I as origin cut the sphere in Q, R and the dividing surface in S, then $2(IS)^{3-n}=(IQ)^{3-n}+(IR)^{3-n}$ except when $n=3$. The results given follow at once.

Ex. 8. If the Earth were made up of two homogeneous solid hemispheres of densities ν, σ, the plane of separation coinciding with the equator, then show that the deviation of the plumb-line from the zenith at any point of the equator would be $\tan^{-1}\left(\frac{2}{\pi}\cdot\frac{\nu\sim\sigma}{\nu+\sigma}\right)$. [St John's Coll., 1882.]

Ex. 9. If a homogeneous solid hemisphere of radius a and density ρ be referred to the centre of the complete sphere as origin, the bounding plane circle as plane of xy and the radius of the hemisphere perpendicular to the plane of xy as axis of z, then the attraction at the origin is along the axis of z and is equal to $\pi\rho a$, where the law of attraction is that of gravitation.

Further show that if V be the potential at a point xyz near the origin, then

$$V=\pi\rho a^2+\pi\rho a z-\tfrac{1}{3}\pi\rho\{x^2+y^2+4z^2\} \text{ (within the hemisphere),}$$

and $$V=\pi\rho a^2+\pi\rho a z-\tfrac{1}{3}\pi\rho\{x^2+y^2-2z^2\} \text{ (without the hemisphere).}$$

[St John's Coll., 1886.]

Ex. 10. Find the resultant attraction of a homogeneous globe on an external particle, the law of attraction being the inverse cube. If O be the particle, C the centre, AB a diameter through O, and if $OB=eOA$ and $\mu=$ the attraction of a unit of mass at a unit of distance, prove that its attraction on $O=\frac{1}{2}\pi\mu(OA/OC)^2$.

[Math. Tripos.]

Ex. 11. The potential of a solid hemisphere of radius a and unit density, at an external point P situated on the axis at a distance ξ from the centre is

$$V=\frac{2\pi a^3}{3\xi}\pm\frac{2\pi}{3\xi}\{(\xi^2+a^2)^{\frac{3}{2}}-\xi^3-\tfrac{3}{2}a^2\xi\},$$

the upper or lower sign being taken according as P is on the convex or plane side of the body.

The potential at an internal point may be found by subtracting from the potential of the complete sphere, that of the missing half.

67. *To find the potential of a shell bounded by any two non-intersecting spheres.*

Let A and B be the centres of the spheres, a and b their radii. Let ρ be the density of the attracting matter which fills the space between these spheres.

The potential at any point P is evidently the difference of the potentials of the spheres each regarded as a solid sphere of density ρ. If R, R' be the distances of P from A and B respectively, the potential at P is

$$V=\tfrac{4}{3}\pi\rho\left(\frac{a^3}{R}-\frac{b^3}{R'}\right) \text{ or } \frac{2\pi\rho}{3}(3a^2-3b^2-R^2+R'^2),$$

according as P is outside or inside both spheres. If P lie between the spheres

$$V=\tfrac{2}{3}\pi\rho\left(3a^2-R^2-\frac{2b^3}{R'}\right).$$

68. We may use the same principle *to find the attraction of a shell bounded by two non-intersecting spheres.*

Suppose, for example, that the attracted point lies within both spheres. The force at P is evidently the resultant of two forces, (1) an attraction equal to $\frac{4}{3}\pi\rho\,.\,PA$ acting along PA, and (2) a repulsion equal to $\frac{4}{3}\pi\rho\,.\,BP$ acting along BP. By the triangle of forces, the resultant of these is equal to $\frac{4}{3}\pi\rho\,.\,BA$ acting parallel to BA. Thus *the attraction at all internal points is the same in direction and magnitude.*

The attraction at an external point may be found in the same way.

69. Ex. 1. Two spheres touch at a point O, and the space between is filled with homogeneous attracting matter. Show that, when the radii differ by an infinitely small quantity, the attractions at two external points, one at O and the other at the opposite extremity of the diameter through O, are as $1:5$.

What is the ratio if the points are inside both spheres?

Ex. 2. A thin layer of attracting matter of mass M is placed on a spherical surface of radius a. If the mass per unit of area is proportional to the square of the distance from a given point O on the circumference, prove that the potential at any external point P is $V=\frac{M}{R}\left(1-\frac{a\cos\phi}{3R}\right)$, where R is the distance of P from the centre A of the sphere, and ϕ is the angle R makes with the radius AO.

This layer may be regarded as filling the space between two spheres which touch at O.

Ex. 3. A thin layer of attracting matter is placed on a sphere, and the mass per unit of area is $A+Bx$, where x is referred to the centre as origin. Show that the potential at an external point P whose abscissa is ξ is $V=\frac{4\pi a^2}{R}\left\{A+\frac{aB\xi}{3R}\right\}$.

70. **A theorem of Gauss.** *The mean value of the potential of any attracting system, taken for all points on any spherical surface, is equal to the potential at the centre due to that part of the attracting system which lies outside the sphere plus the quotient of the mass inside the sphere by the radius.*

Let $d\sigma$ be any element of surface of the sphere, V the potential of all the attracting mass at this element. Let M be the mass inside the sphere and M' that outside, and let V_C be the potential of the latter at the centre C. Let a be the radius of the sphere, then we have to prove that $\frac{\int V d\sigma}{4\pi a^2}=V_C+\frac{M}{a}$.

Let m be the mass of any particle of the attracting system, and let it be situated at a point A. Its potential at any point Q of the sphere is therefore m/AQ. The part of the integral $\int V d\sigma$ due to this mass is therefore $\int m d\sigma/AQ$.

The integral $\int\frac{d\sigma}{AQ}$ is evidently the potential at A of a thin stratum placed on the sphere, of unit surface density, and is therefore equal to $\frac{4\pi a^2}{AC}$ or $\frac{4\pi a^2}{a}$ according as the point A is situated outside or inside the sphere.

Taking all the particles of the attracting system, every particle m outside the sphere contributes a term $4\pi a^2 . m/AC$ to the integral $\int V d\sigma$ while every particle m' inside contributes a term $4\pi a^2 . m'/a$. We therefore have $\dfrac{\int V d\sigma}{4\pi a^2} = \Sigma \dfrac{m}{AC} + \dfrac{\Sigma m'}{a}$. Remembering that V_C is the potential of the external mass at the centre of the sphere, the result follows at once.

71. **Heterogeneous Spherical Shells.** The potential of a heterogeneous spherical shell may be found by the help of Laplace's functions more easily than by any other method. Although there are several cases of heterogeneous shells whose attractions may be found by special artifices, it does not seem useful to stop over these when they can all be treated by one comprehensive method. We must however postpone the discussion of this method until after we have reached Laplace's equation. In the meantime there are some general theorems on heterogeneous shells which are independent of Laplace's functions, and to these we shall now turn our attention.

72. *The potential of a thin heterogeneous spherical shell being supposed known at all internal points, it is required to find the potential at all external points.*

Let O be the centre, a the radius of the sphere. Let P, P' be two points on the same radius, one inside and the other outside, such that $OP . OP' = a^2$. Let $OP = r$, $OP' = r'$.

Let Q be any point on the surface, then since $OP . OP' = OQ^2$ the triangles QOP, $P'OQ$ are similar. It follows that the ratio QP/QP' is constant for all points on the sphere, and that this ratio is equal to a/r'.

Let V, V' be the potentials of the whole shell at P, P'. If m be an element of mass at Q, the potentials of m at P and P' are respectively m/QP and m/QP'. Since these have a constant ratio for all positions of Q and all values of m, the potentials V, V' must have the same ratio. We therefore have $V' = V\dfrac{a}{r'}$.

Let (x, y, z), (x', y', z') be the coordinates of P, P' referred to any rectangular axes with O for origin. Then

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = \frac{r}{r'} = \frac{a^2}{r'^2}.$$

If the potential of the shell at any internal point (x, y, z) be

$V=f(x, y, z)$, then the potential V' at any external point (x', y', z') is found by writing a^2x'/r'^2, a^2y'/r'^2, a^2z'/r'^2 for x, y, z respectively, and multiplying the result by a/r'.

It may be noticed that this proof is an application of Thomson's method of inversion, which will be more fully explained further on.

73. If Y, Y' be parallel components perpendicular to OPP' of the attractions at P, P', we may show by differentiation that $Y' = Y\dfrac{a^3}{r'^3}$. When the points P, P' approach indefinitely near to the surface we have $Y' = Y$.

74. Ex. 1. The potential at an internal point of a thin homogeneous shell of radius a being $V=M/a$, find the potential V' at an external point distant r' from the centre.

We have by the rule $V' = V\dfrac{a}{r'} = \dfrac{M}{r'}$.

Ex. 2. Find the potential at an internal point of the shell described in Art. 69, Ex. 2.

75. **A theorem of Stokes.** Let X, X' be the radial components of the attractions at P, P', estimated positively when directed from the centre. Then since $rr'=a^2$,

$$X=\frac{dV}{dr},\quad X'=\frac{dV'}{dr'}=-\frac{dV}{dr}\frac{a^3}{r'^3}-V\frac{a}{r'^2}=-X\frac{a^3}{r'^3}-V\frac{a}{r'^2},$$

when the points P, P' approach indefinitely near to the surface $r'=a$, and this equation reduces to

$$X'+X=-\frac{V}{a}.$$

We therefore have the following theorem. *The sum of the inward normal attractions at two points on the same radius, one just inside and the other just outside a thin heterogeneous spherical shell, is equal to the potential at either point divided by the radius.* This theorem is given by Sir G. Stokes in his article on the *Figure of the Earth*, and is there proved by the use of Laplace's functions.

Ex. If X, X' be the outward attractions of a thin heterogeneous spherical shell at two points on the same radius at distances r, r' from the centre, V, V' the potentials at the same points, and if $rr'=a^2$ where a is the radius, then

$$Xr^{\frac{3}{2}}+X'r'^{\frac{3}{2}}=-(aVV')^{\frac{1}{2}}.$$

Laplace's, Poisson's and Gauss' theorems.

76. **Laplace's Theorem.** Let (ξ, η, ζ) be the coordinates of any particle A of the attracting matter, and let m be the mass of that particle. Let (x, y, z) be the coordinates of any point P. Taking the particle m apart from the rest of the matter, its potential at P is $V_1=\dfrac{m}{r}$,

where $$r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2 \quad\text{.................(1).}$$

Since $$r\frac{dr}{dx}=x-\xi,$$

we find $$\frac{dV_1}{dx}=-m\frac{x-\xi}{r^3}, \quad \therefore \frac{d^2V_1}{dx^2}=-\frac{m}{r^3}+\frac{3m(x-\xi)^2}{r^5}.$$

In the same way $$\frac{d^2V_1}{dy^2}=-\frac{m}{r^3}+\frac{3m(y-\eta)^2}{r^5},$$
$$\frac{d^2V_1}{dz^2}=-\frac{m}{r^3}+\frac{3m(z-\zeta)^2}{r^5}.$$

Adding up these three expressions and remembering equation (1) we find $$\frac{d^2V_1}{dx^2}+\frac{d^2V_1}{dy^2}+\frac{d^2V_1}{dz^2}=0.$$

Let now V be the potential of the whole attracting matter at P. Then, since V is the sum of the potentials of the several particles, it immediately follows that $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0.$

In this investigation we have assumed that the point P does not coincide with any one of the attracting particles. If it did the meaning of the potential of that particle would require some further consideration. *The theorem has therefore been proved to be true only for a point external to the attracting matter.* It will be presently shown that the right-hand side is not zero when the attracted particle forms a part of the attracting mass.

Laplace's equation is a differential equation which must be satisfied by the potential of every body at all points not occupied by attracting matter. If a general solution of the equation could be found, that solution would comprise within its compass the potential and therefore the component attractions of all bodies.

Laplace's function $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}$ is often written in the abbreviated form ∇^2V.

77. Ex. 1. If the law of attraction between two particles is the inverse nth power of the distance we know by Art. 35 that $V=\frac{1}{n-1}\Sigma\frac{m}{r^{n-1}}+C$. Prove that V satisfies the differential equation $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=(n-2)\Sigma\frac{m}{r^{n+1}}$.

Hence show that the potential cannot be constant throughout any space unoccupied by matter unless the law of attraction is the inverse square. It is assumed that all the m's have the same sign, i.e. that every particle attracts or every particle repels.

Ex. 2. If the law of attraction be as the direct distance, show (1) that $V=-\frac{1}{2}\Sigma mr^2+C$, and (2) that Laplace's equation takes the form $\nabla^2V=-3M$, where M is the attracting mass.

If the law be the inverse distance, (1) $V=-\Sigma m \log r+C$, (2) $\nabla^2 V=-\Sigma \frac{m}{r^2}$.

These results follow easily from first principles, but they may also be deduced from the general theorem given in Ex. 1 by putting $n=\mp 1$.

Ex. 3. A lamina, not necessarily homogeneous, situated in the plane of xy, attracts a point P whose coordinates are (x, y, z). If V_n be the potential when the law of force is the inverse nth power of the distance, and Z_n the component of force in the positive direction of the axis of z, prove

$$V_{n+2}=-\frac{1}{n+1}\cdot\frac{Z_n}{z}, \qquad \nabla^2 V_n=-(n-2)\frac{Z_n}{z}.$$

Show also that the potential of a uniform circular lamina of radius r and situated in the plane of xy, at a point (x, z) in the plane of xz, the origin being at the centre, and the law of force the inverse cube, is

$$V_3=\tfrac{1}{2}\pi\mu\{\log(x^2+z^2+r^2+P)-\log(x^2+z^2-r^2+P)\},$$

where $$P^2=(x^2+z^2+r^2)^2-4x^2r^2.$$

Thence deduce the potential when the law of force is the inverse fifth. *James Roberts' Theorem. Quarterly Journal*, 1881.

Ex. 4. If the potential due to any attracting mass at an external point be V_n when the force attracts according to the inverse nth power of the distance, and X_n be the resolved force in any direction, prove that

$$V_{n+2}=\frac{1}{(n+1)(n-2)}\nabla^2 V_n, \qquad X_{n+2}=\frac{1}{(n+1)(n-2)}\nabla^2 X_n.$$

By this theorem when the potential of a body is known for the law of attraction varying as the inverse distance, the potentials for the laws of the inverse cube, inverse fifth and so on follow by simple differentiation.

78. Laplace's equation is so important in the theory of attraction that we shall frequently have to refer to it not merely in its Cartesian form but also when the coordinates are cylindrical or polar.

Taking cylindrical coordinates first, we put $x=R\cos\phi$, $y=R\sin\phi$, while z remains as the ordinate. We then have

$$\nabla^2 V=\frac{1}{R}\frac{d}{dR}\left(R\frac{dV}{dR}\right)+\frac{1}{R^2}\frac{d^2V}{d\phi^2}+\frac{d^2V}{dz^2}=0.$$

In polar coordinates we have

$$x=r\sin\theta\cos\phi, \quad y=r\sin\theta\sin\phi, \quad z=r\cos\theta;$$

putting $\mu=\cos\theta$ for brevity, we find

$$\nabla^2 V=\frac{1}{r}\frac{d^2}{dr^2}(Vr)+\frac{1}{r^2\sin^2\theta}\frac{d^2V}{d\phi^2}+\frac{1}{r^2}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dV}{d\mu}\right\}=0.$$

These transformations are given in books on the differential calculus and need not be repeated here. The method was simplified by Mr A. Smith in the *Cambridge Math. Journal*, Vol. I. See also Gregory's *Examples on the Differential and Integral Calculus*, Williamson's *Treatise on the Differential Calculus* &c.

Another important theorem should be noticed. If we transform the coordinates from one system of rectangular Cartesian axes x, y, z to another x', y', z', we find

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=\frac{d^2V}{dx'^2}+\frac{d^2V}{dy'^2}+\frac{d^2V}{dz'^2}.$$

79. **Potential at an internal point.** The potential at a point P of any particles situated at the points A_1, A_2, &c. has already been defined in Art. 31

to be $\Sigma \frac{m}{r}$. It is evident from this definition that, if a *finite quantity* of matter be situated at any one of the points A_1, A_2, &c. in a condensed form, the potential at a point P in the immediate neighbourhood of that point is very great, and at that point itself this definition would make the potential infinite. But if the attracting matter is so distributed in space that the mass which occupies any elementary volume dv is ρdv where ρ is finite, we may show that the potential in this portion of space need not be infinite.

Let P be any point in the interior of a mass whose density ρ is constant. Taking P as an origin, let us describe any small surface enclosing P such that every radius vector is positive and equal to $\epsilon f(\theta, \phi)$, where f is any function of the polar angular coordinates θ, ϕ, and ϵ is a small constant factor. An element dv of the volume of this elementary surface distant r from P is equal to $r^2 d\omega dr$, where $d\omega$ is the solid angle subtended at P. When expressed in terms of θ and ϕ, $d\omega$ is equal to $\sin\theta d\theta d\phi$. If then V_2 be the potential at P of the matter filling this surface, we have

$$V_2 = \int \frac{\rho dv}{r} = \iint \rho d\omega r dr \quad\text{.........................(1)},$$

where the limits of integration for r are 0 and $\epsilon f(\theta, \phi)$. It is evident therefore that V_2 is of the order ϵ^2.

It follows that when ϵ is evanescent the value of V_2 is zero. Thus the matter filling the surface may be removed without altering the potential of the whole attracting mass. In finding therefore the potential of a body at any internal point P we may regard P as situated in an infinitely small cavity, and determine the potential as if P were an external point.

Let us consider next the *resolved attraction* at the point P of the matter filling the small surface described above. Let X_2 be the component parallel to the axis of x, then

$$X_2 = \int \frac{\rho dv}{r^2} \cos\theta = \iint \rho \cos\theta d\omega dr \quad\text{.........................(2)},$$

where θ is the angle the radius vector r makes with the axis of x. It is evident that X_2 is of the order ϵ of small quantities, and therefore vanishes when the size of the surface is evanescent.

To simplify the integrations let us suppose that the surface is spherical, so that we may use the formula for the potential already obtained in Art. 65. Let the radius of the sphere be ϵ, let the coordinates of its centre be (a, b, c) and those of P be (x, y, z). Then

$$V_2 = \tfrac{2}{3}\pi\rho\,\{3\epsilon^2 - (x-a)^2 - (y-b)^2 - (z-c)^2\} \quad\text{...................(3)}.$$

It follows at once that

$$\frac{dV_2}{dx} = -\frac{4\pi\rho}{3}(x-a), \qquad \frac{d^2V_2}{dx^2} = -\frac{4\pi\rho}{3} \quad\text{......................(4)}.$$

Since $x-a$ is less than ϵ, it is clear that dV_2/dx is a small quantity of at least the order ϵ, and vanishes when ϵ is evanescent. In the same way the first differential coefficients of V_2 with regard to y and z are evanescent with ϵ. The second differential coefficients of V_2 with regard to x, y or z are however not small.

We have supposed the density of the matter within the evanescent sphere to be uniform. It is however clear that, if we substituted for ρ an expression of the form

$$\rho = \rho_0 + A(x-a) + \&c.$$

we should merely add to the expression for V_2 terms of the order ϵ^3.

If the small cavity is cylindrical we can deduce the values of dV_2/dx and d^2V_2/dx^2 from the expression for the attraction found in Art. 23, x being measured

along the axis of the cylinder. We easily find that dV_2/dx is zero, and that $\frac{d^2V_2}{dx^2} = -4\pi\rho\left(1-\frac{h}{l}\right)$ at points near the centre, where $2h$ is the altitude of the cylinder and l the distance of the centre from any point of either rim. In a flat cylindrical cavity h is small compared with l and this is nearly true at all points on the axis within the cylinder. In a long cylindrical cavity the radius is small compared with the altitude, and the value of the second differential coefficient is zero except when P is close to either end.

Let V and V_1 be the potentials at P of the whole body and of the part of the body outside the small surface enclosing P. Let X and X_1 be the corresponding attractions, then
$$V = V_1 + V_2, \quad X = X_1 + X_2 \qquad (5).$$

Since P is external to the part of the body whose potential is V_1, we have $X_1 = \frac{dV_1}{dx}$.

We have just proved that X_2 and $\frac{dV_2}{dx}$ are both zero when the surface is evanescent.

It immediately follows by differentiating (5) that $X = \frac{dV}{dx}$. Thus *the relation between the resolved attraction and the first differential coefficient of the potential, which has been proved to hold for an external point* (Art. 33), *holds also for an internal point.*

Differentiating (5) a second time, we find for a spherical cavity
$$\frac{d^2V}{dx^2} = \frac{d^2V_1}{dx^2} - \tfrac{4}{3}\pi\rho,$$
with similar relations for the differential coefficients with regard to y and z.

Summing up, we conclude that *the matter in the immediate neighbourhood of any point P supplies nothing to the values of V, dV/dx and X at that point. These values are the same as if P were situated in an evanescent cavity.* This is not necessarily true for the second differential coefficients.

It follows from this that when the point P passes from external space into the interior of a body of finite density both the potential and the attraction undergo no sudden change of magnitude, but the second differential coefficients of the potential are discontinuous in value.

When P traverses an indefinitely thin stratum whose mass per unit of area is finite, the density ρ is not finite. In this case the attraction also may undergo a sudden change of value, the magnitude of which will be considered a little further on.

80. **Poisson's Theorem.** If V be the potential of a body at an internal point P at which the density ρ is finite, then
$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi\rho.$$

Describe a spherical surface of radius ϵ enclosing the point P, let (a, b, c) be the coordinates of its centre, (x, y, z) those of P. Let the radius ϵ be so small that the matter enclosed by the sphere may be regarded as of uniform density.

Let V_2 be the potential at P of the matter within the sphere, V_1 that of the rest of the body, then $V = V_1 + V_2$. But by Laplace's theorem $\nabla^2 V_1 = 0$, hence

$$\nabla^2 V = \nabla^2 V_2 = \frac{d^2 V_2}{dx^2} + \frac{d^2 V_2}{dy^2} + \frac{d^2 V_2}{dz^2}$$
$$= \frac{dX_2}{dx} + \frac{dY_2}{dy} + \frac{dZ_2}{dz},$$

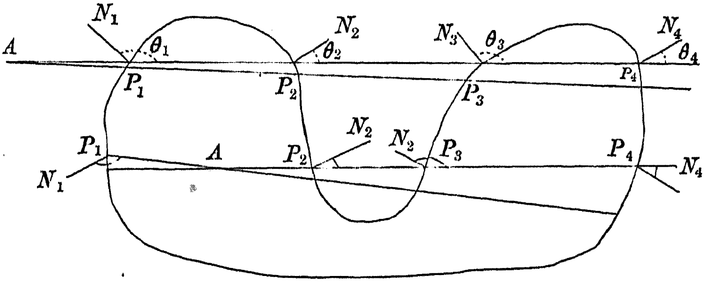
here X_2, Y_2, Z_2 are the resolved attractions at P of the matter ithin the sphere. But by Art. 64

$$X_2 = -\tfrac{4}{3}\pi\rho\,(x-a), \quad Y_2 = -\tfrac{4}{3}\pi\rho\,(y-b), \text{ \&c.}$$

easily follows by substitution that $\nabla^2 V = -4\pi\rho$. Another ·oof of this theorem founded on Gauss' theorem is given a little rther on.

We may notice that the centre of the sphere, though arbitrary in position, must t be taken coincident with P. The reason is that we differentiate V_2 with regard the coordinates of P, i.e. we make P travel from the point (x, y, z) to a neighbour- point $(x+dx, \text{\&c.})$. But since the centre of the sphere is fixed, it cannot be ade to coincide with both the positions of P.

81. **Gauss' Theorem.** *Let S be any closed surface, and let M_1 be the sum of the attracting masses which lie within the surface,*



*the sum of the masses outside. Let $d\sigma$ be any element of area of s surface, F the normal resolute at this element of the attraction the whole mass both internal and external. Then $\int F d\sigma = \pm 4\pi M_1$ ere the integration extends over the whole surface of S and the per or lower sign is taken according as F is estimated positive or yative when the normal force acts inwards**.

Let m be the mass of any particle of the attracting system, and it be situated at the point A. A straight line drawn through to intersect the surface S in any point will also intersect it in ne other point, but, if the surface is re-entrant, it may enter and

* This theorem was given by Gauss in 1839, his paper is translated in Vol. III. Taylor's *Scientific Memoirs*. It was also given by Sir W. Thomson in 1842 in papers on *Electrostatics and Magnetism*. The demonstration given by Sir G. kes in 1849 has been followed here. See his *Mathematical and Physical pers.*

issue from the surface four, six or any even number of times. Let the points of intersection, taken in order, be P_1, P_2, &c., and let the direction P_1P_2, &c. be called the positive direction of the straight line.

Let θ_1, θ_2, &c. be the angles the positive direction of the straight line makes with the normals P_1N_1, P_2N_2, &c. drawn *outwards*. It is evident that where the line enters the surface $\cos\theta$ is negative, and where it issues from the surface $\cos\theta$ is positive, thus the angles θ_1, θ_2, &c. are alternately acute and obtuse.

With A for vertex describe about this straight line an elementary cone whose solid angle is $d\omega$, and let it intersect the surface S in the elementary areas $d\sigma_1$, $d\sigma_2$, &c. If the distances $AP_1 = r_1$, $AP_2 = r_2$, &c., these elementary areas by Art. 25 are

$$d\sigma_1 = r_1^2 d\omega \sec(\pi - \theta_1), \quad d\sigma_2 = r_2^2 d\omega \sec\theta_2, \text{ \&c.} \ldots\ldots (1).$$

If the point A is external to the surface as in the upper part of the figure, the normal resolutes taken positively when acting outwards are

$$F_1 = \frac{m_1}{r_1^2}\cos(\pi - \theta_1), \quad F_2 = -\frac{m}{r_2^2}\cos\theta_2, \text{ \&c.} \ldots\ldots (2).$$

Since the signs of these terms are alternately positive and negative, it follows that when A is external

$$F_1 d\sigma_1 + F_2 d\sigma_2 + \text{\&c.} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

If the point A is internal and lies between P_1 and P_2, as represented in the lower part of the figure, the sign of the force F_1 must be changed. We therefore have

$$F_1 d\sigma_1 + F_2 d\sigma_2 + \text{\&c.} = -2m d\omega \ldots\ldots\ldots\ldots\ldots (4).$$

If the point A lie between P_2 and P_3, the signs of the first two terms in the series (2) are changed, and the equation (4) resumes the form (3), and so on.

If we now let the straight line AP_1P_2 &c. revolve round A into all positions, all the elements of the surface will be included in the integration. We therefore find for an external point

$$\int F d\sigma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

For an internal point the integration of the right-hand side of (4) is limited to a hemisphere of the unit sphere, Art. 25. We therefore have

$$\int F d\sigma = -4\pi m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Let now the system consist of any number of particles m_1, m_2,

&c. inside, and m_1', m_2', &c. outside the surface S. The particles outside contribute nothing to the integral $\int F d\sigma$, while the particles inside contribute respectively $-4\pi m_1$, $-4\pi m_2$, &c. On the whole, when F is measured positively outwards, we have

$$\int F d\sigma = -4\pi M_1 \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

where M_1 stands for the sum of the internal particles m_1, m_2, &c.

The truth of the theorem is not affected if some of the matter, instead of being attractive, be repulsive. Such matter must however be regarded as having a negative mass.

82. The product $F d\sigma$ represents the product of the normal resolute of the attraction at an element multiplied by the area of the element across which it is supposed to act. This product is sometimes called the *flux* or *flow* of the attraction across the elementary area $d\sigma$ in the direction in which the component F is measured. When the particles of the body attract, the proposition asserts that the whole inward flux across any closed surface is equal to 4π multiplied by the mass inside. The product $F d\sigma$ is also called the *induction through the element;* see Maxwell's *Electricity.*

83. *To deduce Poisson's theorem from Gauss' theorem.*

Let dx, dy, dz be the lengths of the sides of a rectangular element having its faces parallel to the coordinate planes. Let the boundary of this element be taken as the surface S. If V be the potential at the centre (x, y, z) the inward flux $F d\sigma$ of the attraction across the two faces parallel to the plane yz are respectively

$$\left(\frac{dV}{dx} - \frac{d^2V}{dx^2}\frac{dx}{2}\right) dy\,dz, \qquad -\left(\frac{dV}{dx} + \frac{d^2V}{dx^2}\frac{dx}{2}\right) dy\,dz.$$

The total flux for these two faces is therefore $-\dfrac{d^2V}{dx^2}dx\,dy\,dz$. In the same way the flux across the other faces parallel to the planes xz, xy are $-\dfrac{d^2V}{dy^2}dx\,dy\,dz$ and $-\dfrac{d^2V}{dz^2}dx\,dy\,dz$. The total mass inside the element is $\rho\,dx\,dy\,dz$. Gauss' theorem gives at once after division by $dx\,dy\,dz$, $-\nabla^2 V = 4\pi\rho$. Stokes, *Cambridge and Dublin Math. J.* 1849.

84. Ex. 1. Deduce from Gauss' theorem the forms of Poisson's theorems as given in Art. 37 for cylindrical and polar coordinates.

Ex. 2. Deduce Poisson's theorem for Cartesian coordinates from Laplace's theorem as in Art. 80, but taking the cavity to be a right circular cylinder and the point P near the centre of gravity.

Theorems on the Potential.

85. *The potential of any attracting system cannot be an absolute maximum or minimum at any point unoccupied by matter*.*

If V be the value of the potential at any point P whose coordinates are x, y, z, the value V' of the potential at any neighbouring point P' whose coordinates are $x+\xi$, $y+\eta$, $z+\zeta$ will be given by

$$V' = V + V_x\xi + V_y\eta + V_z\zeta$$
$$+\tfrac{1}{2}(V_{xx}\xi^2 + V_{yy}\eta^2 + V_{zz}\zeta^2 + 2V_{xy}\xi\eta + 2V_{yz}\eta\zeta + 2V_{zx}\zeta\xi) + \&c.,$$

where partial differential coefficients are represented as usual by suffixes.

If V were a maximum or minimum at the point x, y, z, the first differential coefficients V_x, V_y, V_z would each be zero, and the three second differential coefficients V_{xx}, V_{yy}, V_{zz} (besides fulfilling some other conditions) would have the same sign. But since the point P is unoccupied by matter, they must satisfy Laplace's equations, Art. 76. Their sum must therefore be zero. It is therefore impossible that all three should have the same sign.

We have here assumed that we may apply Taylor's theorem to the potential. That we may do so follows from the definition given in Art. 31. It is clear that the potential at P of a single particle and therefore of a system of particles whose total mass is finite is a function of the coordinates of P which is continuous and finite as long as P does not traverse any attracting matter. We may however put the argument into another form which has the advantage of avoiding the use of series.

86. *Another proof.* With P as centre describe a sphere of small radius. If the potential V were an absolute maximum at P the potential at any point Q of the sphere must be less than that at P. Thus V is decreasing for a displacement along every radius of the sphere. It follows from Art. 33 that the outward normal force F at Q is negative at every point of the sphere. But by Gauss' theorem $\int F d\sigma = 0$, (Art. 81), which requires that F should be positive for some elements of the sphere and negative for

* The theorems in this section may for the most part be found in Gauss' memoir on *Forces varying inversely as the square of the distance*, 1840. In the *Cambridge and Dublin Mathematical Journal*, Vol. IV. 1849, there is an interesting collection of theorems on the potential by Sir G. Stokes. Most of these were already known, but the proofs were much improved and put into new and better forms. This paper is reprinted in his collected works Vol. I. p. 104. The reader may also refer to papers by Lord Kelvin in various volumes of the *Cambridge and Dublin Mathematical Journal*, 1842 and 1843, reprinted in his *Electricity and Magnetism.* There is also a memoir by Chasles in the additions to the *Connaissances des Temps* for 1845.

others. In the same way it may be shown that the potential cannot be an absolute minimum at P.

87. If any arbitrary curve is drawn in space not intersecting any portion of the attracting matter, the potential may vary from point to point of the curve. At some points the potential may be a maximum and at others a minimum for displacements restricted to that curve. What we have proved is that the potential cannot be a maximum or minimum at any point for displacements in every direction.

88. Ex. If the potential is a maximum at a point P not occupied by matter for displacements in two directions at right angles, prove that it must be a minimum for displacements in a direction perpendicular to both.

Taking the coordinate axes parallel to these directions, the result follows at once from Laplace's theorem.

89. *If the potential is equal to any given constant quantity A at all points of a closed surface S which does not contain any portion of the attracting mass, it must be constant and equal to A at all points of the space contained within the surface S.*

For if it were not constant, there would be some point at which either it is greater than at all the other points or less than at all other points. But this has just been proved to be impossible.

90. Ex. 1. As an example of this theorem consider the case of a spherical shell of uniform thickness and density. Describe a concentric sphere within the shell. By symmetry the potential must be the same at all points of its surface. Since there is no attracting matter within this sphere, it follows that the potential is constant throughout its interior.

Ex. 2. If the potential is not constant throughout the superficies of any closed surface S, let A be the greatest and B the least value. Prove that the potential at all points within S lies between A and B. [Stokes.]

Ex. 3. A level surface S completely encloses all the attracting matter of a system. If the consecutive level surfaces extending from S to infinity be drawn, prove that the potential continually decreases outwards from each to the next until it vanishes at an infinite distance.

91. *If the potential is constant throughout any finite space, it is also constant throughout all external space which can be reached without passing through any portion of the attracting mass.* [Stokes.]

This theorem follows from the principle of continuity, but it may also be proved in the following manner.

The external boundary of the space is necessarily a level surface. If possible let A be a point outside the space at which the potential is a little greater than within the space. Since the level surface

through A cannot cut the boundary, the potential at all points in the neighbourhood of A is greater than within the space. We can therefore describe an indefinitely small sphere, passing through A and having its centre O within the space, such that the potential is increasing outwards along every radius drawn from O to any point on the sphere outside the space and is constant along every radius which lies wholly within the space. It follows that the normal force has the same sign at every element of this sphere. This however by Gauss' theorem is impossible. In the same way it may be shown that no point A can exist in the neighbourhood of the space at which the potential is less than within the space.

92. Ex. If the potential is not constant throughout the superficies of any space void of matter, prove that it cannot be constant throughout any finite portion of that space. It may be constant over a surface, but such a surface cannot be closed but must abut on the superficies of the space.

93. **Points of equilibrium.**

If an isolated particle placed at any point P be in equilibrium under the attraction of any system, that point is called a point of equilibrium. When every point of a curve is a point of equilibrium, the curve is called a line or curve of equilibrium.

When the potential of the attracting mass is known, the positions of the points of equilibrium are found by equating the first differential coefficients of the potential to zero, viz. dV/dx, dV/dy, &c.; for these represent the resolved parts of the forces parallel to the axes.

94. *The equilibrium of a free isolated particle cannot be stable for all displacements or unstable for all displacements, but must be stable with reference to some displacements and unstable with reference to others.* Earnshaw's theorem. *Camb. Transac.*, 1839.

If the equilibrium were stable when the particle occupied a position P, the potential must decrease in all directions from P, i.e. the potential would be an absolute maximum at P, which has been proved impossible. In the same way the equilibrium could not be unstable for all displacements.

95. *A particle is in equilibrium at a point P. It is required to find the equation of the cone which, having its vertex at P, separates the displacements for which the equilibrium is stable from those for which it is unstable.*

The level surface which passes through any given point has in general a tangent plane at that point, but when the given point is

a point of equilibrium, such as P, the first differential coefficients V_x, V_y and V_z are zero, and the equation of the plane is nugatory.

Resuming the expression for the potential V' at any point $(x+\xi$, &c.) neighbouring to (x, y, z), we have, Art. 85,

$$V' - V = \tfrac{1}{2} V_{xx}\xi^2 + \&c. + V_{xy}\xi\eta + \&c. + \text{cubes}\ldots\ldots(1).$$

For any small displacement from P which makes V' greater than V, the force on the particle will act from P, and the equilibrium will therefore be unstable (Art. 33). For any displacement from P which makes V' less than V, the equilibrium at P will be stable. To find the directions which separate the stable and unstable displacements, we put $V' = V$. The equation of the *separating cone* is therefore found by equating to zero the terms of the lowest order on the right side of equation (1).

The separating cone is therefore a quadric cone, unless all the differential coefficients of the second order are also zero. It is a real cone, since by Laplace's theorem V_{xx}, V_{yy} and V_{zz} cannot all have the same sign whatever rectangular axes it may be referred to.

It is therefore evident that at a point of equilibrium the level surface has a tangent cone, and that this cone separates the stable and unstable directions of displacement.

96. Ex. 1. Show that three straight lines at right angles can always be drawn through the vertex on the surface of the separating cone. There is an infinite number of such systems of straight lines.

Ex. 2. The level surfaces in the immediate neighbourhood of a point P unoccupied by matter are in general planes, but if P be a position of equilibrium, they are hyperboloids with the separating cone for a common asymptotic cone. If PQ be any radius vector of one of these hyperboloids, the force of restitution for a given small displacement along PQ varies inversely as PQ.

Ex. 3. The lines of force in the immediate neighbourhood of a point of equilibrium, when referred to the principal diameters of the separating cone as axes, are $z^c = Mx^a = Ny^b$, where a, b, c are the reciprocals of V_{xx}, V_{yy}, V_{zz} at the point of equilibrium, and M, N are two arbitrary constants.

Ex. 4. If a number of mutually repelling particles are enclosed in a rigid boundary, show that when in stable equilibrium they all reside on the surface.

[Lord Kelvin.]

Ex. 5. Three thin rods AB, BC, CA, which form a triangle, attract a particle P placed at the centre of the inscribed circle. The particle is therefore in equilibrium. Show that the equilibrium is unstable for all displacements in the plane of the triangle.

97. If two sheets of a level surface intersect along a line, every point of that line is a point of equilibrium.

Let P be such a point, then at least three tangents can be drawn to the sheets of the level surface not all lying in one plane and making finite angles with each other. Since the force along each of these is zero, it follows that the particle is in equilibrium.

98. At every point of the curve of intersection of two sheets of a level surface, the tangent cone becomes two planes which are the tangent planes to the two sheets. The tangent cone may therefore be written in the form

$$(a\xi + b\eta + c\zeta)(a'\xi + b'\eta + c'\zeta) = 0.$$

Comparing this with the form already found, we have

$$aa' + bb' + cc' = V_{xx} + V_{yy} + V_{zz}.$$

This is zero by Laplace's theorem; the tangent planes are therefore at right angles. We therefore infer that, *if two sheets of a level surface intersect, they intersect at right angles.*

99. Ex. 1. The tangent cone becomes two planes whenever its discriminant is zero, but in a level surface these planes cannot be imaginary.

If it were possible, the cone could be reduced to the form

$$(a\xi + b\eta + c\zeta)^2 + (a'\xi + b'\eta + c'\zeta)^2 = 0.$$

This would make $a^2 + a'^2 + b^2 + \&c. = 0$, by Laplace's theorem, which is impossible.

Ex. 2. Show that an isolated line in free space cannot form part of a level surface.

If the potential at a point P were greater than that at some neighbouring point Q and less than that at R, it would follow from the principle of continuity that there must be some point between Q and R on every path from one to the other at which the potential is equal to that at P. If then an isolated line form part of a level surface, the potential must be either greater than at all neighbouring points not on the line or less than at all such points. On either alternative the second proof, by which it is shown that the potential cannot be an absolute maximum or minimum, is contradicted, Art. 86.

100. **Rankine's Theorem.** If at any point of a level surface all the differential coefficients of V up to the nth inclusive with regard to x, y and z are zero, we know from solid geometry that there is a tangent cone of the $(n+1)$th order at that point. If $(n+1)$ sheets intersect along a line, the same thing will be true at every point of that line, and the tangent cone will be the product of the $(n+1)$ tangent planes.

Let us suppose that the level surface is such that at two consecutive points P, P' all the differential coefficients of V up to the nth are zero; let us examine the form of the surface in the immediate neighbourhood of those two points.

Taking P for origin and PP' for the axis of z, we have at the origin all the following differential coefficients equal to zero:

$$\left.\begin{array}{l} \dfrac{d^nV}{dx^n}, \quad \dfrac{d^nV}{dx^{n-1}dy}, \dots \dfrac{d^nV}{dy^n} \\ \dfrac{d^nV}{dx^{n-1}dz}, \quad \dfrac{d^nV}{dy^{n-1}dz}, \quad \dfrac{d^nV}{dx^{n-2}dz^2}, \ \&c. \end{array}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (1).$$

These are also zero when z receives an increment dz; hence their differential coeffi-

cients with regard to z are all zero. It therefore follows that every differential coefficient of V of the $(n+1)$th order which has dz, dz^2, &c. in the denominator is zero at the origin. If therefore V' be the value of the potential at a point ξ, η, ζ, we find on making the expansion by Taylor's theorem

$$\left.\begin{array}{r}V' - V = A_0\xi^{n+1} + A_1\xi^n\eta + \ldots + A_{n+1}\eta^{n+1}\\ + \text{powers of } \xi, \eta, \zeta \text{ of } (n+2)\text{th order}\end{array}\right\} \quad\ldots\ldots\ldots\ldots(2),$$

where A_0, A_1, &c. are constants. It follows that the terms of the lowest order in the expansion do not contain ζ.

The level surface which passes through the origin is given by $V' - V = 0$. This level surface has therefore $(n+1)$ tangent planes at the origin given by

$$U = A_0\xi^{n+1} + A_1\xi^n\eta + \ldots + A_{n+1}\eta^{n+1} = 0 \quad\ldots\ldots\ldots(3).$$

All these tangent planes pass through the two given consecutive points P, P'.

We shall now prove that all these tangent planes are real, and that each makes the same angle with the next in order. The expression for V' given in (2) must satisfy Laplace's equation, hence the expression for U given in (3) must also satisfy that equation. Transforming to cylindrical coordinates, U becomes $U = Pr^{n+1}$, where P is some function of ϕ. By Art. 78, since z is absent from U, we have

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} + \frac{1}{r^2}\frac{d^2U}{d\phi^2} = 0.$$

Substituting, we find

$$\{(n+1)\,n + n + 1\}\,P + \frac{d^2P}{d\phi^2} = 0.$$

$$\therefore\ P = A\cos\{(n+1)\,\phi + \alpha\}.$$

The equation (3) therefore reduces to $\cos\{(n+1)\,\phi + \alpha\} = 0$, which gives $n+1$ planes, making equal angles each with the next in order. The theorem that the tangent planes at any point of a nodal line are inclined at equal angles is due to Rankine.

101. **Tubes of force.** If we draw a line of force through every point of a closed curve, we construct a tube which is called a *tube of force.* By choosing the closed curve properly we can make the section of the tube indefinitely small; it is then called a *filament.* It is evident that the resultant attraction at any point P of a filament acts in the direction of the tangent to the length of the filament.

102. *The magnitude of the attractive force at any point of the same filament is inversely proportional to the area of the normal section of the filament at that point.*

Let σ be the area of the normal section at any point P of the filament, F the attractive force. Consider the portion of the filament bounded by the section at P and that at a neighbouring point Q. Since the filament contains no attracting matter, the total flow of the attraction across the perimeter of this portion is zero. The flow across the sides of the tube is evidently zero, because the resultant force acts along the length of the tube. The total flow across the sections must therefore be zero, hence

$$-F\sigma + \{F\sigma + d\,(F\sigma)\} = 0;$$

$$\therefore\ d\,(F\sigma) = 0.$$

It immediately follows that $F\sigma$ is constant along the whole length of the tube.

103. As an example of this theorem, let the attracting body be a sphere. The lines of force are by symmetry normals to the surface; the filaments are therefore conical surfaces of small angle. If r be the distance of P from the centre, $\sigma = r^2 d\omega$; hence Fr^2 is constant along any line of force. Thus it follows at once that the force of attraction at any external point varies inversely as the square of its distance from the centre.

104. *If two different bodies have equal potentials over the surface of any space not including any attracting matter, they have equal potentials throughout that space, and also at all external space which can be reached without passing through any of the attracting matter of either body.*

For let the attraction of one of the bodies be changed into repulsion. Then the potential due to both bodies is zero over the surface of the given space. That is, the united potential is *constant* over the surface; it is therefore also constant and zero throughout the enclosed space, and at all points of external space which can be reached without crossing any attracting matter; Arts. 89 and 91.

Returning then to the original supposition that both the bodies attract, it easily follows that their potentials are equal.

105. *If two different bodies have equal potentials over the whole boundary of any surface enclosing both, they have equal potentials throughout all external space.*

As before, changing the attraction of one body into repulsion, let us consider the potential of both bodies regarded as one system. Their united potential is therefore zero over the whole boundary of the surface. It is also zero over the boundary of an infinite sphere. Since the space between the surface and the sphere contains no attracting matter, the potential is also zero throughout that space, Art. 89. Returning to the original supposition, that both bodies attract, we see that their potentials must be equal.

106. *If two different bodies have the same level surfaces throughout any empty space, their potentials throughout that space are connected by a linear relation.*

Let V and V' be the two potentials. Since when V is constant, V' is also constant, it follows that V' is some function of V, say $V' = f(V)$. Then by differentiation we easily find

$$\frac{d^2V'}{dx^2} + \frac{d^2V'}{dy^2} + \frac{d^2V'}{dz^2} = \frac{df}{dV}\left\{\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}\right\} + \frac{d^2f}{dV^2}\left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\}.$$

Since the space is external to both bodies, this, by Laplace's equation, reduces to $0 = \dfrac{d^2f}{dV^2}$, unless V is constant throughout the space considered.

This gives $V'=AV+B$, where A and B are two constants. Suppose the space considered includes the points at infinity, then when the attracting masses are finite in size and density both V and V' vanish at such points. We then have $B=0$. Again V and V' must vanish at infinity in the ratio of the attracting masses; we therefore find $V'/V=M'/M$ if M, M' be the masses of the attracting systems. We thus have the theorem; *if two finite bodies have the same external level surfaces and have equal masses, their attractions at all external points are the same in magnitude and direction.* See a paper by the author in the *Quarterly Journal of Mathematics*, 1867.

When the space in which the two bodies have the same level surfaces encloses both bodies, this theorem follows at once from that proved in Art. 105. Since the two bodies have the innermost level surface common, we can by altering the mass of one of them make their potentials equal over that surface. The potentials of the changed bodies are then equal over all external space and the potentials of the original bodies have a constant ratio.

107. As an example of this theorem, consider the case of a spherical shell. The external level surfaces of such a shell and those of an equal mass placed at its centre are both spheres. Hence the attraction of a spherical shell at any external point is the same as that of an equal mass placed at its centre.

Again, the level surfaces of two equal and parallel infinite plates are both planes. Hence their attractions at any point are in a constant ratio. But at an infinite distance the attractions of two such plates when separated by a finite interval tend to equality, hence the ratio of the attractions is unity. It follows that the attraction of an infinite plate at an external point is independent of its distance. In the same way the attraction of an infinite circular cylinder is the same as if the whole mass were uniformly distributed along the axis.

108. The theorems in this section have been enunciated with special reference to the potential of an attracting system, but a little consideration will show that they have a more extended application.

If V be any continuous function which satisfies Laplace's equation and is not infinite within any given space, it follows from the argument in Art. 85 that V cannot be an absolute maximum or minimum at any point within that space.

Most of the other theorems are simple corollaries from this one general principle, and apply therefore to any finite continuous function which satisfies Laplace's equation.

For example, if such a function be constant over the boundary of any space and not infinite within that space, it must be constant throughout that space.

To take another example, let V be a finite continuous function which satisfies Laplace's equation, then $V=c$ is a system of surfaces. If any member of this system intersects itself in a singular line the two sheets are at right angles. If several sheets intersect in a singular line, each tangent plane makes the same angle with the next in order.

Let V, V' be two continuous solutions which are both finite at all points of space bounded by a surface S and are equal at every point of that surface, then they are equal throughout that space. The space considered may be external to S provided the functions are also equal at all points on the surface of some sphere of infinite radius enclosing S. This theorem shows that when the values of a function V are known at all points of the boundary of a space, it is determinate throughout that space, provided it is known to satisfy Laplace's equation and to be finite throughout that space.

109. **Potential at a distant point.** *To find the potential of a body finite in all directions at any distant external point*.*

Let the origin O be a point not far from the body. Let Q be the position of any particle of the body, m its mass, (x, y, z) its coordinates, r its distance from the origin. Let (ξ, η, ζ) be the coordinates of the point P, $OP = r'$, and the angle $POQ = \theta$.

To generalize the investigation we shall assume that the law of attraction is the inverse nth power of the distance. We then have

$$V = \frac{1}{n-1}\Sigma \frac{m}{(r'^2 - 2rr'\cos\theta + r^2)^{\frac{n-1}{2}}}$$

$$= \Sigma \frac{m}{r'^{n-1}}\left\{\frac{1}{n-1} + \frac{r\cos\theta}{r'} + \frac{(n+1)\cos^2\theta - 1}{2}\left(\frac{r}{r'}\right)^2 + \dots\right\}.$$

The first term of the series is $\frac{\Sigma m}{r'^{n-1}}\,\frac{1}{n-1}$. Hence *the attraction at a very distant point is ultimately the same as if the whole mass were collected into a single particle and placed at O.*

To make this a closer approximation to the true attraction, the point O must be such that the second term of the series vanishes. This requires that $\Sigma mr\cos\theta = 0$. Since $rr'\cos\theta = x\xi + y\eta + z\zeta$, this gives $\xi\Sigma mx + \eta\Sigma my + \zeta\Sigma mz = 0$ for all values of ξ, η, ζ. *The point O must therefore be the centre of gravity of the body.*

We have now to consider the third term of the series. Let A, B, C be the moments of inertia of the body about any three straight lines at right angles meeting in O, I the moment of inertia about the straight line OP, then

$$2\Sigma mr^2 = A + B + C, \quad I = \Sigma m(r\sin\theta)^2.$$

* The expansion of the potential at a distant point is originally due to Poisson, but was put into a convenient form by MacCullagh, *R. Irish Trans.* 1855. Some of the following theorems were given by the author in the *Quarterly J.* 1857. The name centrobaric is due to Lord Kelvin, who gave several theorems on these bodies in the *Proc. R. S. E.* 1864. The results in Arts. 115, 116 are taken from Thomson and Tait, 1883.

Writing $1-\sin^2\theta$ for $\cos^2\theta$ and making these substitutions the third term becomes

$$\Sigma m \frac{n(A+B+C)-2(n+1)I}{4} \frac{1}{r'^{n+1}}.$$

When the law of force is the inverse square and the centre of gravity is the origin we arrive at MacCullagh's expression for the potential, viz.

$$V=\frac{M}{r'}+\frac{A+B+C-3I}{2r'^3}+\ldots,$$

where M is the mass of the body.

110. Ex. 1. If two bodies exert equal attractions at all external points, prove that their centres of gravity must coincide and their masses must be equal. The principal axes at their common centre of gravity must coincide in direction and the differences of their moments of inertia about any straight line must be constant.

Ex. 2. When the law of attraction is as the inverse distance, the potential of a single particle takes the form $C-m\log r'$. Prove that the potential of a body at a distant point is

$$V=C-M\log r'+\frac{A+B+C-4I}{4r'^2}+\ldots.$$

111. **Centrobaric bodies.** When a body is such that the direction of its attraction at every point P passes through a point O fixed in the body, the body is said to be *centrobaric*.

It follows from this definition that the potential at P is such a function of the polar coordinates (r', θ', ϕ') of P that the resultant force is dV/dr', the transverse forces $dV/r'd\theta'$, $dV/r'\sin\theta' d\phi'$ being zero. The potential is therefore a function of r' only.

Assuming the law of attraction to be the inverse nth power of the distance, the potential at P is by Art. 109

$$V=\Sigma\frac{m}{r'^{n-1}}\left\{\frac{1}{n-1}+\frac{r\cos\theta}{r'}+\frac{(n+1)\cos^2\theta-1}{2}\left(\frac{r}{r'}\right)^2+\ldots\right\}$$

except when $n=1$.

Since V is a function of r' only, it follows that when the point P is moved about into all positions the coefficients of the several powers of r' remain constant.

If M be the mass of the body, the first term is $\dfrac{1}{n-1}\dfrac{M}{r'^{n-1}}$, *which is the same as if the whole mass were collected at O.*

If (λ, μ, ν) be the direction cosines of OP, the coefficient of $1/r'^n$ is $\Sigma mr\cos\theta=\lambda\Sigma mx+\mu\Sigma my+\nu\Sigma mz$. This cannot be constant unless $\Sigma mx=0$, $\Sigma my=0$, $\Sigma mz=0$; i.e. *the point O must be the centre of gravity.*

The coefficient of the next power, viz. $1/r'^{n+1}$, cannot be constant unless $(n+1)S$ is constant, where

$$S = \Sigma m (r \cos \theta)^2 = \Sigma m (x\lambda + y\mu + z\nu)^2$$
$$= \lambda^2 \Sigma m x^2 + \mu^2 \Sigma m y^2 + \nu^2 \Sigma m z^2 + 2\lambda\mu \Sigma m xy + 2\mu\nu \Sigma m yz + 2\nu\lambda \Sigma m zx.$$

This expression for S cannot be constant* for all values of λ, μ, ν, subject to the condition $\lambda^2 + \mu^2 + \nu^2 = 1$ unless

$$\Sigma m x^2 = \Sigma m y^2 = \Sigma m z^2,$$
$$\Sigma m xy = 0, \ \Sigma m yz = 0, \ \Sigma m zx = 0.$$

It follows from these conditions that the coordinate axes are principal axes of inertia at O. Since these are arbitrary, every straight line through O is a principal axis. It also follows that the moment of inertia about every straight line through O is the same. *The body therefore cannot be centrobaric unless every axis at the centre of gravity is a principal axis.*

If however $n = -1$ these conditions are not necessary. When n has this value the law of attraction is as the direct distance. In this case it has already been proved that a body, whatever be its form, attracts any point as if it were collected into its centre of gravity.

112. Supposing every axis through the centre of gravity to be a principal axis of inertia and the origin to be at the centre of gravity, the expression for V becomes

$$V = \frac{1}{n-1} \frac{M}{r'^{n-1}} + \frac{(n-2) I}{4r'^{n+1}} + \ldots,$$

where I is the moment of inertia about any axis through the centre of gravity.

It appears that this series cannot reduce to the first term unless $n = 2$ or $I = 0$. This latter condition cannot be satisfied unless the masses of some of the particles are negative, i.e. unless some particles attract and others repel P. *Assuming that all the particles attract P, we see that the attraction of a body cannot be the same as if its whole mass were collected into its centre of gravity unless the law of force be either as the direct distance or as the inverse square.*

113. Ex. If the law of force be the inverse square, the potential of a body at all external points cannot be the same as that of two masses M_1 and M_2 placed at

* If this is not obvious, place the point P on the axis of z; then $\lambda = 0$, $\mu = 0$ and $S = \Sigma m z^2$. In the same way $S = \Sigma m y^2$ and $S = \Sigma m z^2$. The equation then reduces to $\lambda\mu \Sigma m xy + \mu\nu \Sigma m yz + \nu\lambda \Sigma m xy = 0$. Putting $\lambda = 0$, this proves that $\Sigma m yz = 0$. Similarly $\Sigma m zx = 0$, $\Sigma m xy = 0$.

two points A, B fixed in the body unless (1) the body and masses have their centres of gravity coincident, (2) the moments of inertia of the body about every axis through the centre of gravity perpendicular to AB are equal. See Ex. 1, Art. 110.

114. **Potential constant in a cavity.** *In a similar manner, when a body has a cavity within its substance we may determine the necessary conditions that the potential should be constant throughout the cavity.*

Taking the origin within the cavity, we have at all points close to the origin

$$V = \Sigma \frac{m}{r^{n-1}} \left\{ \frac{1}{n-1} + \frac{r' \cos\theta}{r} + \frac{(n+1)\cos^2\theta - 1}{2} \left(\frac{r'}{r}\right)^2 + \dots \right\},$$

expanding in powers of r'/r because r' is less than r.

This cannot be independent of r' unless the coefficient of each power of r' is zero. Equating the coefficient of r'^2 to zero, we have

$$\Sigma \frac{m}{r^{n+1}} \{(n+1)\cos^2\theta - 1\} = 0.$$

Writing α, β, γ for $\Sigma \dfrac{mx^2}{r^{n+3}}$, $\Sigma \dfrac{my^2}{r^{n+3}}$, $\Sigma \dfrac{mz^2}{r^{n+3}}$ and putting the point P in succession on the axes of x, y, z we have $n\alpha = \beta + \gamma$, $n\beta = \gamma + \alpha$, $n\gamma = \alpha + \beta$. These equations cannot coexist unless $n = 2$ or α, β, γ are each zero. The latter alternative requires that all the m's should not have the same sign. Hence *if every particle of the body be attractive, the potential cannot be constant throughout any cavity unless the law of attraction is the inverse square.* See Art. 77, Ex. 1.

115. Assuming that a body attracts all points in external space as if the whole mass were collected into its centre of gravity, prove that (1) the centre of gravity is inside the external boundary, (2) the external boundary is a single closed surface.

If the centre of gravity O were in the same external space as the attracted point P, we could surround it by a small sphere, centre O, radius ϵ, which does not enclose any particle of the attracting mass. The flux across this sphere is therefore zero, Art. 81. But since the force on P tends always to O, the flux is also $4\pi M$. These results contradict each other unless the whole mass is equal to zero.

Again, if the attracting system consist of two separate portions, the centre of gravity O must lie inside one of them. Enclosing the other portion in a sphere, the flux across the surface is $4\pi M'$, if M' be the mass of this portion. But since O lies outside the sphere, it is also zero. These results cannot coexist unless the mass of that portion is zero.

116. A body B is such that the resultant attraction between it and a given body A is a force which always passes through the centre of gravity O of B, in whatever position A is placed. Prove that the resultant attraction between B and every body is a force which passes through the centre of gravity of B.

Let the body A be turned about a fixed point P sufficiently distant from B, that

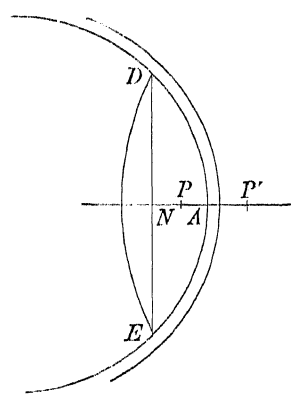
the body A in its motion never meets the fixed body B. In all these positions the resultant attraction of A on B is a force which passes through the centre of gravity of B. Hence if every particle of the mass of A be uniformly distributed over the surface of the sphere which that particle describes in its motions, the resultant attraction of the mass thus obtained is also a force which passes through the centre of gravity of B. The mass thus obtained is a spherical shell whose resultant attraction at any point of B is the same as if it were collected at the centre P. The resultant action between the body B and a particle placed at P is a force which passes both through P and the centre of gravity of B. The body B is therefore centrobaric for all points P beyond a certain distance and therefore for all points of space which can be reached from P without passing over any of the attracting mass, Art. 104.

Attraction of a thin stratum.

117. **A theorem due to Green.** Let a thin heterogeneous stratum of attracting matter be placed on a surface which has no conical points or other singularities. Let ρ be the density and t the thickness at any point A of the surface, and let $m = \rho t$, so that m is the surface density at the point A. In what follows we shall regard m as finite and t as indefinitely small, so that ρ is very large.

Let P, P' be two points situated on the normal at A, one inside the surface and the other outside, both close to the stratum; it is required to find the attractions at P and P'.

Draw a plane, parallel to the tangent plane at A, and cutting the normal in a point N such that AN is indefinitely small, but not indefinitely smaller than the thickness of the stratum at A. This plane intersects the surface in a small closed curve DE which is ultimately a conic, and every diameter of it is infinitely greater than either the abscissa AN or the thickness t. This plane divides the whole attracting stratum into two parts whose attractions at P, P' will be separately considered.



Let us consider first the attraction of the small adjacent portion DAE. Since both PA and $P'A$ are indefinitely smaller than any of its diameters, this portion of the stratum bears to either P or P' the same geometrical relation that an infinitely extended plate of infinite radius of curvature does to a point at a finite distance. If we apply the same method to find the attraction of this portion

that we used to find that of an infinite plate, we divide each into corresponding elements and arrive at the same integrals taken between the same limits. The results are therefore the same. The attractions of the adjacent portion of the stratum at P and P' are therefore independent of the distances of those points from A (provided only they are indefinitely small), and are each equal to $2\pi m$. These attractions are also directed along the normal to the stratum but in opposite directions. Their difference is therefore $4\pi m$.

Consider next the attraction of the portion of the stratum remote from A. Since the distance PP' is infinitely smaller than the distance of either P or P' from the nearest attracting element, the components in any given direction of the attractions at those points can differ only by terms of the order mt. Thus if mF and mF' are the components parallel to the axis of x, the difference is $mt\,(dF/dx)$.

Taking both portions of the attracting stratum into the account, we see that the difference of the normal components of the attractions at P and P' differs from $4\pi m$ by quantities of the order mt. Representing these components by X and X', we have, since the stratum is indefinitely thin, $X' - X = 4\pi m$.

118. We may also show that the parallel tangential components of attraction just inside and outside the stratum are equal.

Let the axis of y be parallel to a tangent at C to either boundary of the stratum. Let Y, Y' be the components of attraction at P, P'. Considering first the adjacent portion DE of the stratum, it has already been shown that the resultant attractions at P, P' are each directed along the normal PP'; hence this portion contributes nothing to Y or Y'. Considering next the remote portion of the stratum, it has been shown that the components Y, Y' differ by terms of the order mt. In the limit therefore when t is very thin, we have $Y' = Y$.

119. We shall now show that the potentials at P, P' are also equal. The potentials due to the remote portion of the stratum for the same reasons as before can differ only by terms of the order mt. Consider next the portion of the stratum adjacent to A; the potentials at two points equally distant from the two faces of the stratum evidently differ by terms of an order higher than mt. See also Art. 63, Ex. 2.

Taking both portions of the stratum, we see that the potentials at P and P' are ultimately equal.

120. It follows from this proposition that *if a point travel from a position P just within a thin stratum to another P' just outside, both on the same normal, the normal component of the attraction is increased by the quantity $4\pi m$, where m is the surface density. At the same time the tangential components of the attraction and the potential are unaltered.*

The theorem that $X' - X = 4\pi m$ is of great importance in the theory of attraction. It is commonly called Green's theorem. It was afterwards rediscovered by Gauss in 1840. The mode of proof followed above was given by Lord Kelvin in 1842, see the reprint of his papers on *Electrostatics and Magnetism*, Art. 7. See also Thomson and Tait, Art. 478.

121. We may also deduce Green's theorem from the proposition, due to Gauss, that the flux of the attraction over a closed surface is 4π multiplied by the mass inside. See Art. 81.

Let the axis of x be a normal to the stratum, measured positively inwards, and let it cut the boundaries in the points A, A'. Let us consider the flux of the attraction across an element of volume whose edges parallel to the axes x, y, z are respectively AA', dy and dz.

Let X, X'; Y, Y'; Z, Z' be the normal components of the attraction at the six faces of the element, let t be the thickness AA', and ρ the density of the stratum at A. We then have

$$(X' - X)\,dy\,dz + (Y' - Y)\,t\,dz + (Z' - Z)\,t\,dy = 4\pi\rho t\,dy\,dz.$$

We shall now suppose that the *surface density* of the stratum is finite and equal to m, then $\rho t = m$.

Consider first the two faces perpendicular to the axis of y; since there is attracting matter of *continuous density* on both sides of each of these faces, the attractions Y, Y' differ by a quantity of the order dy, i.e. $Y' - Y = \frac{dY}{dy}dy$. In the same way

$$Z' - Z = \frac{dZ}{dz}dz.$$

Consider next the two faces perpendicular to the axis of x; since there is attracting matter on one side only of each face, the normal attractions X and X' differ by a quantity which is infinitely larger than either $Y' - Y$ or $Z' - Z$.

Substituting in the above equation and dividing by $dy\,dz$, we have

$$X' - X + \frac{dY}{dy}t + \frac{dZ}{dz}t = 4\pi m.$$

In the limit, when the stratum is indefinitely thin, we have

$$X' - X = 4\pi m.$$

122. Ex. 1. A thin layer of heterogeneous attracting matter is placed on a sphere of radius a. If V be the potential and m the surface density at any point A, show that the normal attractions on each side of the stratum are $V/2a \pm 2\pi m$. See Art. 75.

Ex. 2. Prove that, if matter attracting according to the law of the inverse square be so distributed over a closed surface that the resultant attraction on every external particle in the immediate neighbourhood is in the direction of the normal, the resultant attraction on every internal point is zero.

The outer boundary of the stratum is by definition a level surface. The inner boundary is therefore also a level surface. The result then follows from Art. 89 because there is no attracting matter within that surface.

123. **Green's equivalent stratum.** Let S be a closed level surface, or a closed portion of a level surface, of some real fixed system of particles situated partly within and partly without S. Let M be that portion of the system which is within S, M' the portion outside. Let V be the potential at S of the whole system.

It is required to find the law of density of a thin stratum placed on the surface S, such that its potential together with that of M' shall be equal to V at all points of S.

That such a stratum exists is evident, because, by building up the stratum particle by particle we can make the potential at every point of S to vary from $-\infty$ to $+\infty$ *. Another proof of this is given in the next section.

Since the sum of the potentials of the stratum and M' is equal to that of M and M' at all points of S, it follows that the potential of the stratum is the same as that of M. Thus the stratum and M have equal potentials at all points of a surface just outside S and zero potentials at all points of a sphere of infinite radius enclosing S, and neither system has any attracting matter between S and the sphere. It follows that the potential of the stratum is equal to that of M at all external points, Art. 105. The attractions of the stratum and M are therefore also equal in magnitude at all external points and act in the same direction.

Since the potentials of the stratum and M are equal at all points external to S, they must vanish in a ratio of equality at an infinite distance. It follows at once that the mass of the stratum is equal to M.

Since the sum of the potentials of the stratum and M' is constant at all points of a surface just inside S, and no particle of either the stratum or of M' lies within this surface, it follows that the sum of the potentials is constant throughout the

* Removing the portion M, let a thin stratum formed of mutually repelling particles be placed on the surface S, and let us suppose that each particle is free to move without impediment along the surface, but not to leave it. When these have assumed a position of equilibrium under the influence of their own repulsions and of that of M', the resultant force on each particle must act in the direction normal to S. This being true at every point of S, it follows that S is a level surface of the stratum and M'.

In some positions of equilibrium the stratum may be collected together at special points of the surface. These may even be the stable positions if the force between the particles is attractive instead of repulsive. We discuss here only those positions of equilibrium in which the surface is entirely covered.

interior of the surface S. The attractions therefore of the stratum and of M' are equal in magnitude, and act in opposite directions.

Lastly, if ρ be the surface density of the stratum, F and F' the inward normal attractions just inside and just outside, we have by a theorem of Green $4\pi\rho = F' - F$. But it has just been shown that F' is the same as the normal attraction of M, and $-F$ is the normal attraction of M'. The sum of the normal attractions of M and M' is the normal attraction of the whole fixed system; hence we have $\rho = -\frac{1}{4\pi}\frac{dV}{dn}$, where dn is an element of the normal drawn outwards. Thus ρ is known when the normal force of the fixed system at every point of S is known.

Since this gives only one value of ρ, it follows that there is but one stratum which can satisfy the given conditions.

124. Summing up these results we have a theorem due to Green.

Let there be a system of attracting particles whose potential at every point is known. Let S be a closed portion of a level surface, and let V be the potential at S. Let M be the portion of the system inside S, M' the portion outside. Let a thin stratum of attracting particles be placed on S such that its surface density ρ is given by $\rho = -\frac{1}{4\pi}\frac{dV}{dn}$, where dn is an element of the normal drawn outward. Then

(1) The resultant of the attractions of the stratum and M' on any one of its particles is normal to S.

(2) The potential of the stratum at all points external to S is the same as that of M.

(3) The sum of the potential of the stratum and that of M' is constant at all points internal to S and is equal to V.

(4) The mass of the stratum is equal to M.

These results are of primary importance in the theory of Electricity. When a body of any form is electrified, it is shown that the electricity exhibits itself on the surface of the body as if it were a collection of particles so arranged that the potential is constant throughout the interior of the body, the intensity of the electricity at any point being measured by the density of the stratum at that point. It follows as a result of Green's theorem that, if a system can be found such that the surface of the electrified body is one of its level surfaces, the law of distribution of the electricity can be immediately deduced. The density ρ at any point P of the stratum is given by equating $4\pi\rho$ to the inward normal force of the system at P.

125. Ex. 1. If the surface S enclose all the fixed attracting particles, so that $M'=0$, prove that all the particles of the stratum are pressed by their mutual attractions inwards against the constraining surface.

Ex. 2. Prove that the equivalent stratum and the portion M of the fixed system enclosed by the surface S have

(1) Their centres of gravity coincident.

(2) The directions of their principal axes at the centre of gravity coincident.

(3) The difference of their moments of inertia about any straight line through the centre of gravity constant, Art. 113.

Ex. 3. V_0 is an equipotential surface wholly surrounding the attracting mass A, and V_1 is another outside V_0. The space between is filled up with matter, the density at any point being $R^2 f(V)$, where R is the whole attraction of A at the point and V is the potential. Find the potential at any point external to V_1 in terms of the potential of A at that point. Also show how to find the force at any point between V_0 and V_1. [St John's Coll.]

Ex. 4. A solid homogeneous body of unit density attracts an external point P according to the law of nature. Let (r', θ', ϕ') be the coordinates of any point Q on the surface of the body, R the distance of Q from P, r the distance of P from the origin. If V be the potential at P, prove

$$-r^3 \frac{d}{dr}\left(\frac{V}{r^2}\right) = \iint \frac{r'^3 \sin\theta' d\theta' d\phi'}{R}.$$

This theorem is due to Ivory, *Phil. Trans.* 1824. See Todhunter's *History of Attractions*, &c., Art. 1424.

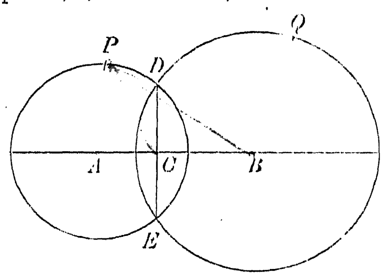
Ex. 5. A thin layer of attracting matter is laid on the surface of a homogeneous attracting body. If the density of the body be ρ and the surface density of the layer be ρp, where p is the perpendicular on the tangent plane drawn from any origin O, prove that the potential V' of the thin layer at any external point P is given by the equation $V' = \rho \int \frac{p\, d\sigma}{R} = 2V - r\frac{dV}{dr}$, where V is the potential of the solid body at P and r is the distance of P from the assumed origin O.

Ex. 6. A homogeneous solid of unit density differs so little from a sphere that the square of the excess of the radius over the radius of the sphere may be neglected, and V is its potential at an external point P so near the surface that the square of the distance also may be neglected. If r be the radius vector of the point P measured from any origin near the centre of the sphere, and a the radius of the sphere, prove that $-a\frac{dV}{dr} = \frac{2\pi ar}{3} + \frac{1}{2}V$. [Laplace's Theorem.]

Treat the homogeneous sphere and the thin layer as separate bodies. Taking the layer first, let A_1 and V_1 be its attraction and potential at P. We find by Art. 122, Ex. 1, that $2A_1 - V_1/a = 4\pi(r-a)$. Taking the sphere next, let A_2 and V_2 be its attraction and potential at P; we find $2A_2 - V_2/a = \frac{4}{3}\pi\{a - 3(r-a)\}$, since $r - a$ is small. Adding the two we obtain Laplace's theorem.

126. Ex. 1. Two spheres intersect at right angles; it is required to distribute on the bounding surface a thin stratum of matter whose potential at any internal point shall be constant.

Let A, B be the centres of the two spheres, a, b their radii; and let AB cut the plane of intersection in C. Let DE be the circle in which the spheres intersect. Let P be any point on the sphere whose centre is A, Q any point on the other sphere.



To determine the density of the stratum we must first discover a system such that the boundary of the two spheres is one of its level surfaces. Since

$$AC \cdot AB = AP^2,$$

the triangles PAC, PAB are similar;

hence
$$-\frac{1}{\sqrt{AB}}\frac{1}{CP}+\frac{1}{\sqrt{AC}}\frac{1}{BP}=0 \quad\text{.......................}(1).$$
Similarly, since $BC \cdot BA = BQ^2$, we have
$$-\frac{1}{\sqrt{AB}}\frac{1}{CQ}+\frac{1}{\sqrt{BC}}\frac{1}{AQ}=0 \quad\text{...........................}(2).$$
Let three particles, whose masses α, β, γ are such that
$$\alpha\sqrt{CB}=\beta\sqrt{CA}=-\gamma\sqrt{AB}=m \quad\text{.......................}(3),$$
be placed at A, B, C respectively. It is evident from (1) and (2) that the potential of these masses is constant over the surface of each sphere; also since the spheres intersect, this constant is the same for each sphere. The two spheres therefore form a level surface of the three particles. It also follows that the attractive forces of the three particles at any point of the circle DE are in equilibrium. Art. 97.

The equation (3) giving the ratios of α, β, γ may also be written in the form
$$\frac{\alpha}{a}=\frac{\beta}{b}=-\frac{\gamma}{CD} \quad\text{.......................................}(4).$$

If we apply Green's theorem to this system different results may be obtained according to the portions of the two spheres which are chosen to form the boundary. If we take as the level surface the larger portions of the spheres bounded by the plane of intersection, all the three particles at A, B, C are internal. If we take as the boundary the larger portion of the sphere whose centre is A and the smaller portion of the other sphere, the particles at B and C are external and that at A internal, and so on.

The normal attraction F at the point P is
$$F=\frac{\alpha}{a^2}+\frac{\beta}{BP^2}\cos APB+\frac{\gamma}{CP^2}\cos APC.$$
By expressing these cosines in terms of the sides of the triangles APB, APC, and using equation (1) we see that the two last terms reduce to the form L/CP^3, where L is some constant. To find L we notice that $F=0$ at every point of the circle DE,

hence
$$4\pi\rho=F=\frac{\alpha}{a^2}\left\{1-\left(\frac{CD}{CP}\right)^3\right\}.$$
In the same way, if ρ' be the density at Q we have
$$4\pi\rho'=\frac{\beta}{b^2}\left\{1-\left(\frac{CD}{CQ}\right)^3\right\}.$$

The result is that, if the boundary be formed by the larger portions of both spheres, the stratum is such that its potential is constant throughout the interior, and the densities at P and Q are ρ and ρ'. If the boundary be formed by the larger portion of the sphere whose centre is A and the smaller portion of the other sphere the stratum is such that its potential together with the potentials of the external particles at B and C is constant throughout the interior, the densities ρ, ρ' being given by the same expressions as before.

The electrified state of two spheres intersecting orthogonally is given in Maxwell's *Electricity*, Part I. Chap. XI.; other interesting cases are also discussed such as the case of two spheres intersecting at an angle π/n where n is an integer, or that of three spheres intersecting orthogonally.

Ex. 2. An infinite cylinder is placed with its axis parallel to a plane and at a given distance from it. It is required to place a thin stratum on the plane and cylinder such that the potential at any point of the plane shall be zero and that the potential shall be constant throughout the interior of the cylinder. Show that

the density at any point of the cylinder varies inversely as the distance of that point from the plane.

The plane and cylinder are different level surfaces of two uniform infinite rods, one attracting and the other repelling with equal forces. One rod lies inside the cylinder and the other on the side of the plane opposite to the cylinder. Removing the former we coat the sphere with Green's equivalent stratum. Removing the latter we coat the plane with the equivalent stratum.

Ex. 3. An uninsulated conductor consists of a sphere and an infinitely large and infinitely thin plane passing through the centre A of the sphere. If it be exposed to the influence of a given charge of electricity at the point B, where BA is perpendicular to the plane, prove that, C being a point on BA produced such that AC is equal to AB, the superficial density at any point P on the hemispherical surface nearest to B is proportional to $\frac{1}{BP^3} - \frac{1}{CP^3}$. [Math. Tripos, 1877.]

The potential of the electrical stratum at any internal point of an uninsulated conductor is zero.

127. *To find the attraction of any thin heterogeneous stratum on an elementary portion of itself.*

Referring to the figure in Art. 117 let the element be a small cylinder whose base is the indefinitely small area $d\sigma$ situated at A, and whose altitude is the thickness t of the stratum. Dividing the attracting stratum as before into two parts by a plane DE parallel to the tangent plane at A and intersecting the surface of the stratum in a small conic, let the linear dimensions of $d\sigma$ be infinitely smaller than any diameter of this conic.

We shall consider the attractions of these two portions of the stratum separately. The attraction of the adjacent portion on the cylindrical element is evidently zero. The attraction of the remote portion of the stratum at two points P, P' situated one inside and one outside, both being close to the stratum, can differ only by terms of the order mt, where m is the surface density at A; Art. 117. Let F represent the normal component of either of these attractions. The normal attraction of the remote portion of the stratum and therefore of the whole stratum is ultimately equal to $Fmd\sigma$.

Let X, X' represent as before the normal attractions of the whole stratum at the points P, P', then by Art. 117

$$X = F - 2\pi m, \quad X' = F + 2\pi m.$$

$$\therefore F = \tfrac{1}{2}(X + X').$$

Thus the whole normal accelerating force acting on the element is equal to the arithmetic mean of the normal attractions just inside and just outside the stratum.

The reader will find other modes of proof in the reprint of Thomson's papers on Electricity &c. Art. 88, and in Maxwell's *Electricity*, Art. 78.

Green's Theorem.

128. One important case of Green's theorem has been already given in Art. 123. The mode of proof there given, though short, is indirect and depends on several properties of the potential. It has therefore seemed advantageous to give here a direct proof of a somewhat more general theorem, showing, by simple summation, that the attraction of Green's film is the same as that of the body it displaces.

Both these modes of proof depend on the definition of a potential and are adapted to a theory of attraction. Another view of the subject has therefore been added in which the functions considered are perfectly general and need not be potentials of any real system of attracting points.

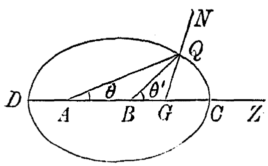
129. **Geometrical Proof.** Let A, B be two fixed points, AQ, BQ radii vectores drawn to any point Q of a surface S. Let $r=AQ$, $r'=BQ$, $AB=c$; let $d\sigma$ be any element of area at Q, dn an element of the normal GQN drawn outwards at Q. Let also

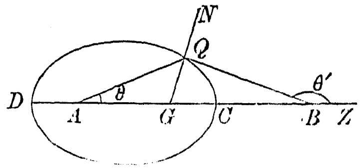
$$U=-\int\frac{d\sigma}{r'}\frac{d}{dn}\left(\frac{1}{r}\right)+\int\frac{d\sigma}{r}\frac{d}{dn}\left(\frac{1}{r'}\right) \qquad\qquad (1),$$

where the integration extends over the whole surface of S.

Let $d\omega$, $d\omega'$ be the solid angles subtended at A and B by the element $d\sigma$, and let ψ, ψ' be the angles the radii vectores AQ, BQ make with the *inward* normal QG, then

$$d\sigma\frac{d}{dn}\left(\frac{1}{r}\right)=-\frac{d\sigma}{r^2}\frac{dr}{dn}=-\frac{d\sigma}{r^2}\cos\psi=-d\omega,$$





with a similar expression for the origin B. Substituting these values we find

$$U=\int\left(\frac{d\omega}{r'}-\frac{d\omega'}{r}\right) \qquad\qquad (2).$$

Let us next take the straight line ABZ as an axis of reference, let θ, θ' be the angles QAZ, QBZ respectively and let ϕ be the angle the plane ABQ makes with some fixed plane passing through AB. We therefore have $d\omega=\sin\theta d\theta d\phi$, with a similar expression for $d\omega'$. Substituting in (2), we find

$$U=\int d\phi\int\left(\frac{\sin\theta}{r'}d\theta-\frac{\sin\theta'}{r}d\theta'\right) \qquad\qquad (3).$$

Let $\chi=\theta'-\theta$, so that χ is the angle AQB. If c be regarded as positive when B is on the right-hand side of A, we have $\frac{\sin\theta}{r'}=\frac{\sin\theta'}{r}=\frac{\sin\chi}{c}$. Substituting again, we have

$$U=-\int d\phi\int\frac{\sin\chi}{c}d\chi=\int\frac{d\phi}{c}[\cos\chi] \qquad\qquad (4).$$

The limits of $\cos\chi$ depend on whether the points A and B are within or without the surface S. If A and B are both inside, then χ varies from zero at C to zero at D. In this case the value of $U=0$. The same result is true if A and B are both outside.

Next, let A be inside and B outside but on the right-hand side of A, then χ varies from π at C to zero at D. If B be outside but on the left-hand side of A, so that c is negative, the limits are zero at C to π at D. Thus $[\cos\chi]=2$ or -2 according as c is positive or negative. It will be convenient to regard c as always positive in order that we may afterwards use the definition of a potential given in Art. 31, we therefore put $[\cos\chi]/c=2/c$. Lastly, if A be outside and B inside, we find in the same way $[\cos\chi]/c=-2/c$ when c is regarded as positive.

Integrating (4) from $\phi=0$ to $\phi=2\pi$, we have, when A and B are on the same side of the surface, $U=0$; when A and B are on opposite sides of the surface $U=4\pi/c$ or $-4\pi/c$ according as B is outside or inside.

Referring back to the definition of U as given by (1), we notice that if the element of normal dn were measured positively inwards, the sign of U would be changed. We may therefore say that when the points A and B are on opposite sides of the surface $U=\dfrac{4\pi}{c}$ or $-\dfrac{4\pi}{c}$ according as B is on the side towards which the normal is measured or on the opposite side.

130. Let A be the seat of a particle of mass m_1, then m_1/r is its potential at any point Q of the surface. If we represent this potential by V_1, we have

$$-\int\frac{d\sigma}{r'}\frac{dV_1}{dn}+\int d\sigma V_1\frac{d}{dn}\left(\frac{1}{r'}\right)=0 \text{ or } \pm 4\pi\frac{m_1}{c_1}\dots\dots(5),$$

where c_1 stands for the distance of the particle m_1 from B and is to be always taken positively.

Let the attracting system consist of any number of particles, m_1, m_2, &c. being on the side of S opposite to B and m_1', m_2', &c. on the same side as B. The latter system of particles contributes nothing to the right-hand side of (5), while the particles of the former system contribute the terms $4\pi m_1/c_1$, $4\pi m_2/c_2$, &c. Adding these together we have

$$-\int\frac{d\sigma}{r'}\frac{dV}{dn}+\int d\sigma V\frac{d}{dn}\frac{1}{r'}=\pm 4\pi V_B\dots\dots(6),$$

where V is the potential at the element $d\sigma$ of all the attracting particles, V_B the potential at B of the particles on the side of S opposite to B, and the upper or lower sign is to be taken according as B lies on the side towards which dn is measured or on the opposite side. Lastly r' is the distance of the element $d\sigma$ from B.

131. The equation (6) admits of considerable simplification when the surface S is a closed portion of any level surface of the attracting system. In this case V is the same at all points of the surface. Representing this constant value of V by V_s, the equation (6) then becomes $-\int\frac{d\sigma}{r'}\frac{dV}{dn}+V_s\int d\sigma\frac{d}{dn}\frac{1}{r'}=\pm 4\pi V_B$. To fix our ideas we suppose that dn is measured outwards.

It has already been shown that $d\sigma\dfrac{d}{dn}\dfrac{1}{r'}=-d\omega'$ and that $\int d\omega'=0$ or 4π according as the point B is outside or inside the closed surface S. We therefore have

$$-\int\frac{d\sigma}{r'}\frac{dV}{dn}-\begin{pmatrix}4\pi V_s\\ \text{or } 0\end{pmatrix}=\mp 4\pi V_B\dots\dots(7),$$

the upper or lower alternative being taken according as B is on the inside or outside the surface.

We may express both the alternatives of this important equation in the form of a theorem.

If a thin layer of attracting matter is placed on a closed portion of a level surface S of any attracting system so that the surface density at any element $d\sigma$ is

$$\rho'=-\frac{1}{4\pi}\frac{dV}{dn}\dots\dots(8),$$

where dn is an element of the normal measured outwards, then (1) *the potential of the layer at any external point B is the same as the potential of that part of the attracting system which lies inside the stratum;*

(2) *the potential of the layer at any internal point B, increased by the potential at the same point of that portion of the attracting system which is external to S, is equal to the potential of the whole attracting system at its level surface S.*

132. The equation (6) helps us to find the potential of all the matter on one side of any arbitrary closed surface S at any point B on the other side when we know both the potential and the normal force of the whole attracting system at all points of that surface.

Supposing dn to be measured towards that side of S on which B lies, we have by (6)

$$4\pi V_B = \int \left(V \frac{d}{dn} \frac{1}{r'} - \frac{1}{r'} \frac{dV}{dn} \right) d\sigma \quad \text{.......... (9)}.$$

$$= -\int \frac{dVr'}{dn} \frac{d\sigma}{r'^2} \quad \text{.......... (10)}.$$

From this equation we deduce the following theorem.

If on any arbitrary closed surface S we place a thin layer of attracting matter whose surface density is given by

$$\rho' = -\frac{1}{4\pi} \frac{1}{r'} \frac{d(Vr')}{dn} \quad \text{.......... (11)},$$

its potential at the point B is the same as the potential of that portion of the attracting system which lies on the side of the surface S opposite to B.

Suppose for example all the attracting system lies on one side of the surface S; then the potential of the thin layer at B is zero when B lies on the same side of S as the attracting matter and is equal to the potential of the attracting system when B lies on the opposite side.

It may be noticed that the density of this layer at any given element of the surface S is a function of the distance of that element from B, and is, therefore, not the same for all positions of B. The density of the layer described in Art. 131 is independent of the position of B, and depends only on the attraction at the given element.

133. The theorem of Gauss given in Art. 81 is a particular case of the theorem expressed by equation (6).

Let the arbitrary point B be taken outside and so far off from S that the distance r' is sensibly the same for all elements of S. We then have $V_B = \frac{M}{r'}$, where M is the mass of that part of the system which lies within S. Also $\frac{d}{dn}\frac{1}{r'} = 0$, so that the equation (6) reduces to

$$-\int d\sigma \frac{dV}{dn} = 4\pi M \quad \text{.......... (12)}.$$

The equation (6) is a particular case of Green's general theorem. Thus it appears that Green's theorem includes the theorems of Gauss, Poisson and Laplace as particular cases.

134. We may deduce a more general theorem from equation (6). Let there be a second system of attracting particles situated, like the first, partly within and partly without the surface S. Let μ_1, μ_2, &c. be the masses of those within, μ_1', μ_2', &c. the masses of those without S. Let their positions in space be denoted by B_1, B_2, &c., B_1', B_2', &c.

Referring now to (6), let the point B be the seat of a particle of mass μ, then μ/r' is its potential at any point Q of the surface; let us represent this potential by V_1'.

Accordingly we have $-\int V_1'\dfrac{dV}{dn}d\sigma+\int V\dfrac{dV_1'}{dn}d\sigma=\pm 4\pi\mu V_B$ ....(13),

the positive or negative sign being taken on the right-hand side according as the point B is on that side of S towards which dn is measured or the opposite.

Repeating this process for every one of the masses of the second system, we obtain for each an equation like (13). Adding these together, we have, when dn is measured outwards,

$$\int\left\{-V'\frac{dV}{dn}+V\frac{dV'}{dn}\right\}d\sigma=-4\pi\Sigma\mu V_B+4\pi\Sigma\mu' V_{B'},$$

where V' is the potential at the element $d\sigma$ of all the attracting particles of the second system, while $\Sigma\mu' V_{B'}$ is the mutual potential of the particles m_1, m_2, &c. of the first system and the particles μ_1', μ_2', &c. of the second, and $\Sigma\mu V_B$ is the mutual potential of the particles m_1', m_2', &c. of the first system and μ_1, μ_2, &c. of the second.

We may put this equation into the form of a theorem. Let there be two systems of particles in presence of each other such that each particle attracts every particle of the other system but does not attract any particle of its own. Let any surface S be drawn enclosing some particles of each system, let V, V' be the potentials of the two systems respectively at any element $d\sigma$ of the surface, dn an element of the normal drawn outwards. Let W' be the mutual potential or work of the inside particles of the first system and the outside particles of the second, W that of the inside particles of the second and the outside particles of the first, then

$$\int d\sigma\left\{V\frac{dV'}{dn}-V'\frac{dV}{dn}\right\}=4\pi(W'-W)\text{.......................(14).}$$

If F, F' be the normal components of force at the element $d\sigma$ outwards due to the two systems respectively, we may write the equation in the form

$$\int d\sigma\,(VF'-V'F)=4\pi\,(W'-W)\text{.........................(15).}$$

If we add to each of the expressions W, W' the mutual work of those particles of the two systems which are inside the surface S, the difference $W-W'$ is clearly unchanged. Thus if W_1 be the mutual work of the first system and the inside particles of the second, W_1' that of the second system and the inside particles of the first, we have $W-W'=W_1-W_1'$.

135. **Analytical Proof.** In the proof which has been given of equation (14) it has been assumed that the functions V, V' are potentials of some real collections of particles, and this is all that is necessary when we are considering the attractions of bodies. But by adopting another mode of proof we may remove this restriction and obtain Green's theorem in another form.

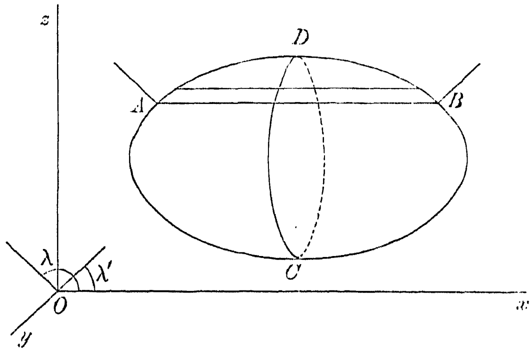
Let any portion of space be enclosed, or separated from the rest of space, by a surface which we shall call S. Supposing V, V' to be any two functions of x, y, z, let us integrate

$$U=\iiint\left(\frac{dV}{dx}\frac{dV'}{dx}+\frac{dV}{dy}\frac{dV'}{dy}+\frac{dV}{dz}\frac{dV'}{dz}\right)dx\,dy\,dz\text{.......(1)}$$

throughout the given space S. Taking the first term we have, by integration by parts,

$$\iint\left[V\frac{dV'}{dx}\right]dy\,dz-\iiint V\frac{d^2V'}{dx^2}dx\,dy\,dz\text{..........(2).}$$

We have here integrated all the elements which lie in a column parallel to the axis of x. Let AB be one of these columns, and let



it intersect the surface S at A and B in the elementary areas $d\sigma$, $d\sigma'$. If (λ', μ', ν') be the direction cosines of the outward normal at the upper limit B, we have $dydz = \lambda' d\sigma'$. In the same way if (λ, μ, ν) be the direction cosines of the outward normal at the limit A, we have $dydz = -\lambda d\sigma$, since λ' is positive and λ negative. The quantity in square brackets in the first term of (2) is to be taken between the limits A and B, and is therefore

$$\left(V\frac{dV'}{dx}\right)_B \lambda' d\sigma' - \left(V\frac{dV'}{dx}\right)_A (-\lambda d\sigma) \quad \text{.........}(3),$$

where the suffix indicates the place at which the value of the quantity in brackets is to be taken. The two terms in (3) have now to be integrated, the first for all elements, such as B, on the right-hand side of the bounding curve CD, and the second for all elements, such as A, on the left. These are together therefore the same as

$$\int V\frac{dV'}{dx}\lambda d\sigma \quad \text{.......................}(4),$$

taken for all elements of the surface, where λ now stands for the cosine of the angle the outward normal at $d\sigma$ makes with the axis of x.

Treating the other terms of (1) in the same way we see that

$$U = \int V\left(\frac{dV'}{dx}\lambda + \frac{dV'}{dy}\mu + \frac{dV'}{dz}\nu\right) - \iiint V\left(\frac{d^2V'}{dx^2} + \frac{d^2V'}{dy^2} + \frac{d^2V'}{dz^2}\right)dxdydz.$$

Let now dn be an element of the normal at $d\sigma$ drawn outwards, then

$$\frac{dV'}{dx}\lambda + \frac{dV'}{dy}\mu + \frac{dV'}{dz}\nu = \frac{dV'}{dn}.$$

Also let ρ, ρ' be such functions of x, y, z that

$$-4\pi\rho' = \nabla^2 V', \qquad -4\pi\rho = \nabla^2 V.$$

Then substituting we find $\qquad U = \int V \frac{dV'}{dn} d\sigma + 4\pi \int V\rho' dv,$
where dv stands for an element of volume.

In the same way we have by interchanging V, V' in the symmetrical expression U $\qquad U = \int V' \frac{dV}{dn} d\sigma + 4\pi \int V'\rho dv.$

Equating these two values of U, we arrive at the following equation, usually called Green's theorem:

$$\int \left(V \frac{dV'}{dn} - V' \frac{dV}{dn} \right) d\sigma = 4\pi \int (V'\rho - V\rho')\, dv \quad \text{......(5).}$$

Applying this equation to the case of attraction as in Art. 134, we see that the two values of U just found may be written in the forms $\quad U = \int VF' d\sigma + 4\pi W_1 = \int V'F d\sigma + 4\pi W_1' \quad$ ....(6).
where, as in the article referred to, V and V' are the potentials of two systems of particles, and W_1 is the work of system I. and that part of system II. which is internal to S, and W_1' is the work of system II. and that part of I. which is internal to S.

136. The expression for U admits also of interpretation. Let (X, Y, Z) (X', Y', Z') be the components of force due to the systems I. and II. respectively at any point (x, y, z) within S. Let R, R' be the resultant forces, ϕ the angle between the directions of R, R'. Then by (1)

$$U = \int (XX' + YY' + ZZ')\, dv = \int RR' \cos\phi\, dv \quad \text{......(7).}$$

If the two systems are the same, i.e. if the particles have equal masses and occupy the same positions in the two systems, the equations (6) reduce to $\qquad U = \int R^2 dv = \int VF d\sigma + 4\pi \int V\rho dv,$
where V is the potential, and F the outward normal force at $d\sigma$.

137. If the functions V, V' satisfy Laplace's equation, we have $\rho = 0$, $\rho' = 0$, the equations (5) then reduce to

$$U = \int V \frac{dV'}{dn} d\sigma = \int V' \frac{dV}{dn} d\sigma \quad \text{................(8).}$$

138. Instead of considering the space internal to S we may integrate through the space on the other side of S. Supposing then that the space under consideration is external to S and that both the products VdV'/dn, $V'dV/dn$ are zero at all points of a sphere of infinite radius enclosing S we find by integrating through the space between these two surfaces

$$\int \left(V \frac{dV'}{dn} - V' \frac{dV}{dn} \right) d\sigma = 4\pi (W_1' - W_1) \quad \text{........(9).}$$

Here however dV/dn differs in sign from its former meaning, for dn has been measured outwards *from* the space over which the integration extends. Also W_1, W_1' are the mutual works of the first and second systems respectively on the parts of the other system within the space of integration.

139. **Two useful cases.** Referring to the general theorem expressed by (5) in Art. 135 let us put for V' unity. Since this value satisfies Laplace's equation we have $\rho' = 0$. The equation then assumes the simpler form

$$\int \frac{dV}{dn} d\sigma = -4\pi \int \rho dv.$$

This is really Gauss' theorem in another form. Here V is any function of (x, y, z) which is continuous and finite, $-4\pi\rho$ is the result of its substitution in Poisson's equation, and the integration extends on the right-hand side throughout any arbitrary space and on the left side over the bounding surface. See Art. 133.

Another useful case is when $V' = 1/r'$, where r' is the distance of any point within the space of integration from some external point P. In this case, as in the last, $\rho' = 0$. The general theorem then reduces to $\int \left\{ V \frac{d}{dn}\left(\frac{1}{r'}\right) - \frac{1}{r'}\frac{dV}{dn} \right\} d\sigma = 4\pi \int \frac{\rho dv}{r'}$ ...... (10),

which agrees with the equation (9) Art. 132 already obtained.

140. **Points at which V is infinite.** If P be any arbitrary point taken in the interior of the space bounded by the surface S, it is evident that one of the columns of integration parallel to each coordinate axis will pass through P. It is necessary that in each of these three columns the subject of integration should be finite. We have therefore assumed in the proof given in Art. 135 that (1) both the functions V, V' are finite and continuous, (2) that their first and second differential coefficients with regard to x, y, z are each finite throughout the space considered. If any of the functions be infinite at some point A within S, we must surround that point by an infinitesimal sphere, and integrate only over the space between the sphere and the surface S.

Let us suppose that, besides other terms, V has a term of the form $\frac{m}{r}$, where r is a distance measured in any direction from the point A. Let the centre of the small sphere be taken at A and let the radius be ϵ. We shall find the values of $\int V \frac{dV'}{dn} d\sigma$ and

$\int V' \frac{dV}{dn} d\sigma$ taken over the surface of the sphere, and $\int V'\rho dv$, $\int V\rho' dv$ taken throughout the volume. Joining these values to the corresponding integrals in Green's equation we arrive at a second equation which is true when the infinity at the point A is ignored and the integrations extend only through the surface S. In these integrations over the surface of the sphere dn is to be measured from the space of integration and therefore towards the centre; it will however be found convenient to measure dn *from* the centre of the sphere and to *subtract* the results of the integrals from the corresponding integrals in Green's equation.

Taking first the terms other than m/r, we notice that since $d\sigma = \epsilon^2 d\omega$ both the integrals taken over the surface of the sphere are of the order ϵ^2, while those taken throughout the volume are of the order ϵ^3. All these therefore vanish when the sphere is indefinitely small.

Taking next the term m/r in the expression for V, we have

$$\int V \frac{dV'}{dn} d\sigma = \int \frac{m}{\epsilon} \frac{dV'}{dn} \epsilon^2 d\omega = 0.$$

Since the sphere is infinitely small the value of V' at any point of its surface is the same as that at its centre and may be written V_A'. We then have

$$\int V' \frac{dV}{dn} d\sigma = V_A' \int - \frac{m}{\epsilon^2} \epsilon^2 d\omega = -4\pi m V_A'.$$

Subtracting these values from the general equation (5), we find

$$\int \left(V \frac{dV'}{dn} - V' \frac{dV}{dn} \right) d\sigma - 4\pi m V_A' = 4\pi \int (V'\rho - V\rho')\, dv,$$

where the integration on the right-hand side extends throughout the space between the infinitely small sphere and the surface S.

Regarding the term m/r in the expression for V to be due to the presence of a mass m at the point A, the term $4\pi m V_A'$ is equal to the value of the right-hand side when the integration extends through the volume of the infinitesimal sphere. To show this we notice that for all elements within the sphere the right-hand side is the same as $4\pi \{V_A' \int \rho dv - \rho' \int (m/r)\, dv\}$. Writing $dv = r^2 d\omega dr$, and integrating from $r = 0$ to ϵ, the value of the second term is zero and that of the first is $4\pi m V_A'$. The term $4\pi m V_A'$ may therefore be transposed to the right-hand side of the

equation, and included in that integral when the integration extends throughout the whole space bounded by the surface S.

Other infinite terms of the form m/r occurring in either V or V' may be treated in the same way. If, as in Art. 134, W_1, W_1' are respectively the mutual works of the systems whose potentials are V, V' and the inside particles of the other system, we have

$$\int\left(V\frac{dV'}{dn}-V'\frac{dV}{dn}\right)d\sigma=4\pi\,(W_1'-W_1)$$

whether these infinite terms exist or not.

141. Ex. If V contain a term m/r^k, show that some of the integrals connected with the infinitely small sphere are infinite if k is greater than unity and are all zero if k is less than unity.

142. **Multiple-valued functions.** It has been supposed in these theorems that the functions V, V' have only one value at the same point of space. If they are potentials of attracting masses, they are each of the form $\Sigma m/r$ and can have only one value. But if they are obtained as solutions of Laplace's equations, as in hydrodynamics, they may be many-valued functions. Thus let a fluid be running round in a ring-like vessel. If V be the velocity potential at any point P, we know by the principles of hydrodynamics that $dV/ds=u$, where s is the arc described, and u is the velocity at P. Since the velocity is always positive, the velocity potential V must always increase as P travels round the ring. When P has made a complete turn, it comes to the point it started from, and V has a different value. If we put Laplace's equation into cylindrical coordinates (Art. 78), we easily see that $V=\tan^{-1}y/x=\phi$ satisfies the equation and represents such a motion.

143. In order to apply Green's equation to a multiple-valued function by integrating throughout the space enclosed in a ring-shaped surface we must deprive the function of its multiple values by placing a barrier at any point and including this barrier as one of the boundaries. In this way the point P is prevented from making a complete circuit and the function is reduced to a single-valued form. It may be that the surface has several ring-like passages interlacing, and it may then be necessary to insert several barriers before the function is reduced to a single-valued form.

Taking the simpler case of a single ring-like surface, let us suppose that the potential V is always increased by the same quantity c when the point P starting from any position has made a complete circuit and has returned to the same position again. Similarly let V' be increased by c'. Let $d\alpha$ be an element of the area of a barrier placed anywhere across the ring-like cavity. Let s be an arc measured from the barrier round the ring to the barrier again, say from $s=0$ to $s=l$. Consider the part of the boundary formed by the two sides of the barrier; remembering that dn is measured outwards, we have $dn=-ds$ for the side defined by $s=0$, and $dn=ds$ for the side $s=l$. We thus have, when we integrate over both sides of the barrier,

$$\int V\frac{dV'}{dn}\,d\alpha=-\int V\frac{dV'}{ds}\,d\alpha+\int(V+c)\frac{d\,(V'+c')}{ds}\,d\alpha=c\int\frac{dV'}{ds}\,d\alpha.$$

Supposing V and V' to be solutions of Laplace's equation, Green's theorem becomes

$$U=\int V\frac{dV'}{dn}\,d\sigma+c\int\frac{dV'}{ds}\,d\alpha=\int V'\frac{dV}{dn}\,d\sigma+c'\int\frac{dV}{ds}\,d\alpha,$$

where along the surface S, dn is measured outwards, and across the barrier ds is measured in the positive direction round the ring.

144. Ex. 1. Let V, V' represent as before any two functions of (x, y, z), and let a be a third finite function of the same variables. Beginning with

$$U = \iiint a^2 \left(\frac{dV}{dx}\frac{dV'}{dx} + \frac{dV}{dy}\frac{dV'}{dy} + \frac{dV}{dz}\frac{dV'}{dz} \right) dxdydz,$$

show by the same succession of integrations as in Art. 140 that

$$U = \int a^2 V' \frac{dV}{dn}\, d\sigma + 4\pi \int V' \rho dv$$

$$= \int a^2 V \frac{dV'}{dn}\, d\sigma + 4\pi \int V \rho' dv,$$

where $$-4\pi\rho = \frac{d}{dx}\left(a^2 \frac{dV}{dx}\right) + \frac{d}{dy}\left(a^2 \frac{dV}{dy}\right) + \frac{d}{dz}\left(a^2 \frac{dV}{dz}\right),$$

and $-4\pi\rho'$ represents a similar expression with V' written for V. This is Lord Kelvin's extension of Green's theorem. See Thomson and Tait, vol. I. p. 167. Edition of 1879.

Ex. 2. If V, V' be both solutions of the differential equation

$$\frac{d}{dx}\left(a^2 \frac{dV}{dx}\right) + \frac{d}{dy}\left(a^2 \frac{dV}{dy}\right) + \frac{d}{dz}\left(a^2 \frac{dV}{dz}\right) = 0,$$

and if also $V = V'$ at all points of a closed surface S, prove that $V = V'$ throughout the enclosed space.

Let $u = V - V'$, then u also is a solution of the differential equation. Writing u for both V, V' in the general theorem of Ex. 1, we have

$$\int a^2 \left\{ \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2 \right\} dv = \int a^2 u \frac{du}{dn}\, d\sigma.$$

The right-hand side is zero since u vanishes at all points of the surface S. But the left-hand side is the sum of a number of positive quantities and cannot be zero unless each vanishes. Thus du/dx, du/dy, du/dz are each zero at all points inside S, i.e. the function u is a constant. Since it is given equal to zero at the surface S, it must be zero at all points within S.

This differential equation is of great importance in the analytical theory of heat.

Ex. 3. Show in the same way that if $\dfrac{dV}{dn} = \dfrac{dV'}{dn}$ at all points of the surface S then $V = V'$ throughout the space enclosed.

Ex. 4. If both V and V', besides being solutions of the differential equation, also satisfy the equation $\dfrac{dV}{dn} = -kV$ at all points of S, where k is a function of the coordinates which is always positive, prove that $V = V'$.

Ex. 5. If V be one solution of the differential equation in Ex. (2) such that $\dfrac{dV}{dn} = -kV$ at all points of a surface S, where k is always positive, prove that there is no other solution of that differential equation which satisfies this condition.

145. **Converse Problem.** Hitherto the body has been supposed given and its potential has been the quantity sought. Conversely, we may require to determine the body when the potential is given. It is evident that Poisson's equation

$$4\pi\rho = -\nabla^2 V \text{ ....(1)}$$

supplies a partial solution to this question. The potential V being given throughout all space, we find ρ by merely performing the proper differentiations. This value of ρ, if finite throughout space, determines the only body which could have the given

potential. If the potential is given as a discontinuous function of the coordinates, difficulties may arise in applying Poisson's equation at the points or surfaces where the discontinuity occurs. The following theorem will therefore be useful.

146. Let the potential V throughout any given portion S of space be the given function $\phi(\xi, \eta, \zeta)$ of the coordinates; throughout a neighbouring space S', let the potential be a function $\psi(\xi, \eta, \zeta)$ and so on, with the condition that where two spaces meet the functions have equal values. We shall then prove that the attracting system is made up of (1) a matter filling each of the spaces S, S', &c. whose volume density is given by $\rho = -\frac{1}{4\pi}\nabla^2 V$, where V is the potential within that space; (2) a superficial stratum surrounding each space whose surface density is given by $\rho' = \frac{1}{4\pi}F$, where F is the normal force just within the boundary, measured positively outwards, so that $F = dV/dn$ where dn is an element of the outward normal.

Let P be the point (ξ, η, ζ) at which the potential is given, let dv be an element of volume of the attracting body at Q, and let r be the distance PQ. Considering the space S the potential at P of the matter within it is $\int \frac{\rho dv}{r}$, the integral extending over all the elements within S and ρ being the volume density given above.

First, let P be outside the surface S. Writing $dxdydz$ for dv, we may effect these integrations by the process employed in proving Green's theorem in Art. 135. But it is unnecessary to repeat the steps, for we have already obtained the result in equation (10) of Art. 139. We have therefore

$$\int \frac{\rho dv}{r} = \frac{1}{4\pi}\int \left\{\phi(x, y, z)\frac{d}{dn}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{d\phi(x, y, z)}{dn}\right\} d\sigma \quad \text{..............(2)},$$

where the integrations on the right-hand side extend over the boundary of the space S. The integral on the right-hand side represents the potential at P of a thin stratum of matter placed on the boundary whose surface density at any point Q is given by

$$D = \frac{1}{4\pi}\left\{r\phi \cdot \frac{d}{dn}\left(\frac{1}{r}\right) - \frac{d\phi}{dn}\right\} \quad \text{........................(3)},$$

where r is the distance PQ. Art. 132, Eq. (11).

Next, let (ξ, η, ζ) or P lie within the space S. We must surround P by a sphere of small radius ϵ and include the surface of this sphere as part of the boundary of the space S, as in Art. 140. Considering the surface of this sphere and remembering that dn is drawn positively towards its centre, the integral (2) becomes

$$\frac{1}{4\pi}\phi(\xi, \eta, \zeta)\int \frac{d\sigma}{\epsilon^2} - \frac{1}{4\pi\epsilon}\int \frac{d\phi(x, y, z)}{dn}d\sigma.$$

The first term is evidently $\phi(\xi, \eta, \zeta)$ and the second term * vanishes by Gauss' theorem, Art. 139. Hence the potential at P of the matter between the sphere and the surface S is equal to that of the stratum on the surface S together with $\phi(\xi, \eta, \zeta)$. The potential at the same point of the matter within the infinitely small sphere is zero by Art. 79.

* We may also prove in an elementary manner that the second integral is zero. Let PN be a normal to the level surface which passes through P and let $PN = d\nu$; let θ be the angle the normal dn to the sphere makes with PN, then $\frac{d\phi}{dn} = -\frac{d\phi}{d\nu}\cos\theta$. The second integral is therefore $\frac{d\phi}{d\nu}\int \cos\theta d\sigma$, where the integration extends over the surface of the sphere. This is evidently zero.

147. The result arrived at is that the potential at P of the matter within the space S, diminished by the potential at the same point of the stratum placed on the boundary of S, is equal to $\phi(\xi, \eta, \zeta)$ or zero according as P is inside or outside S. Supposing this stratum to be included, with the sign of its density changed, as part of the attracting system, the proposed conditions are satisfied for the space S.

Treating the neighbouring space S' in the same way, we obtain an internal density determined as before by Poisson's equation and a superficial density which, when its sign is changed, is the same as that given by (3) except that the function ϕ is replaced by ψ and the element dn of the normal is measured in the opposite direction.

Adding together the two superficial densities and remembering that ϕ and ψ are equal at those points of the boundary which are common to S and S', we observe that the first terms of each destroy each other. We therefore find for the density of the superficial stratum

$$\rho' = \frac{1}{4\pi}\left\{\frac{d}{dn}\phi(x, y, z) + \frac{d}{dn'}\psi(x, y, z)\right\},$$

where dn and dn' are measured outwards from the spaces S and S' respectively, so that $dn = -dn'$. We notice that this law of density is independent of the position of P.

148. Ex. 1. The potential at a point Q is

$$2\pi(b^2 - a^2),\quad \tfrac{2}{3}\pi(3b^2 - r^2 - 2a^3/r)\quad \text{or}\quad \tfrac{4}{3}\pi(b^3 - a^3)/r,$$

according as the distance r of Q from the origin is less than a, lies between a and b, or is greater than b. Find the attracting system.

Considering the space in which r is less than a, we see that both the volume and surface densities are zero.

Considering the space in which r lies between a and b, the volume density is found by substituting in $\rho = -\dfrac{1}{4\pi r}\dfrac{d^2Vr}{dr^2}$, Art. 146, and is equal to unity. The superficial density is found by substituting in $\rho' = \dfrac{1}{4\pi}\dfrac{dV}{dn}$ and is zero at the inner boundary and $-(b^3 - a^3)/3b^2$ at the outer.

Lastly in the space in which r is greater than b, the volume density is zero and the superficial density $(b^3 - a^3)/3b^2$.

Adding all these together, we find that the attracting body is a spherical shell of radii a and b and unit density.

Ex. 2. Find the attracting system whose potential V is equal to

$$\mu(1 - Lx^2 - My^2 - Nz^2)$$

at all points within the ellipsoid $Lx^2 + My^2 + Nz^2 = 1$ and zero at all external points.

The system is a homogeneous ellipsoid whose density is $\mu(L + M + N)/2\pi$, together with a superficial stratum whose surface density at any point Q is $-\dfrac{\mu}{2\pi}\dfrac{1}{p}$ where p is the perpendicular on the tangent plane at Q.

Since this stratum is equivalent to a thin homogeneous confocal shell (see Vol. I. Art. 430), this result supplies a simple relation between the potential of a homogeneous solid ellipsoid and that of a homogeneous confocal shell.

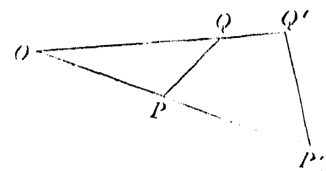
Ex. 3. Let r, r' be radii vectores drawn from two fixed points A, B, the first within and the second without the sphere whose equation is $r' = nr$. Find the attracting system whose potential at any point P is $1/AP$ or n/BP according as P is without or within the sphere.

The system is a stratum on the sphere whose surface density varies inversely as the cube of either r or r'.

Theory of inversion.

149. **Inversion from a point.** Let O be any assumed origin, and let Q be a point moving in any given manner. If on the radius vector OQ we take a point Q' so that $OQ \cdot OQ' = k^2$, then Q and Q' are called inverse points. If Q trace out a curve, Q' traces out the inverse curve; if Q trace out a surface or solid, Q' traces out the inverse surface or solid.

Let P', Q' be the inverse points of P, Q, then since the products $OP \cdot OP'$, $OQ \cdot OQ'$ are equal and the angles POQ, $P'OQ'$ are the same, the triangles POQ, $P'OQ'$ are similar. We therefore have



$$\frac{1}{P'Q'} = \frac{1}{PQ} \cdot \frac{OQ}{OP'} \quad \text{.........(1).}$$

Let m, m' be the masses of two particles placed respectively at Q, Q', and let the densities be such that $m' = m \dfrac{k}{OQ}$ ....(2).

Multiplying equations (1) and (2) together, we see that the potential at P' of m' is equal to that at P of m, after multiplication by a quantity k/OP' which is independent of the position of Q.

Let any number of particles of given masses m_1, m_2, &c. be placed at different points Q_1, Q_2, &c., and let the corresponding masses m_1', m_2', &c. be placed at the inverse points Q_1', Q_2', &c. Then since an equation similar to (2) holds for each pair of masses, we have by addition

$$\left(\begin{array}{c}\text{Potential at } P' \\ \text{of the inverse system}\end{array}\right) = \left(\begin{array}{c}\text{Potential at } P \\ \text{of the given system}\end{array}\right) \frac{k}{OP'} \quad \text{.........(3)}$$

which may be compendiously written $V' = V \dfrac{k}{OP'}$.

150. If the given masses m_1, m_2, &c. are arranged so as to form an arc, surface or solid, the inverse masses will also be arranged in the same ways. It will therefore be necessary to discover some rule by which we can compare the density at any point of the given system with that at the corresponding point of the inverse system.

Using the same figure as before but changing the meaning of P, let PQ now represent any elementary arc of the locus of Q, then $P'Q'$ represents the corresponding inverse arc. If the locus of Q is a curve, we infer from the similarity of the triangles POQ,

$P'OQ'$ that the lengths of the elementary arcs $P'Q'$, PQ are in the ratio OQ'/OP, i.e. OQ'/OQ ultimately. Hence by (2) the ratio of the line densities of the arcs $P'Q'$, PQ is equal to k/OQ'.

If the locus of Q is a surface, the elementary areas $P'Q'$, PQ are in the ratio of the squares of the homologous sides, i.e. as OQ'^2 to OQ^2. Hence by (2) the ratio of the surface densities at Q' and Q is equal to $(k/OQ')^3$.

If Q travel over all points of space enclosed by a surface, the elementary volumes at Q', Q are ultimately in the ratio $OQ'^2 \, d\omega \, . \, d(OQ')$ to $OQ^2 \, d\omega \, . \, d(OQ)$. Since $OQ \, . \, OQ' = k^2$, this ratio is equal to OQ'^3/OQ^3. Hence by (2) the ratio of the densities at Q' and Q is equal to $(k/OQ')^5$.

Summing up these results, we see that

$$\begin{pmatrix}\text{density at } Q' \\ \text{of the inverse system}\end{pmatrix} = \begin{pmatrix}\text{density at } Q \\ \text{of the given system}\end{pmatrix} \cdot \left(\frac{k}{OQ'}\right)^{2d-1} \quad \ldots(4),$$

where d represents the dimensions of the system, i.e. $d = 1$, 2, or 3 according as the system is an arc, a surface or a volume.

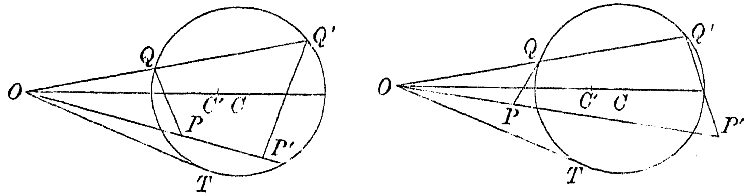
151. Ex. 1. If the law of force be the inverse nth power of the distance, the potential of a particle m takes the form $\dfrac{1}{n-1}\dfrac{m}{r^{n-1}}$. Prove that the equations corresponding to (2), (3), and (4) become

$$m' = m\left(\frac{k}{OQ'}\right)^{1-n}, \qquad \rho' = \rho\left(\frac{k}{OQ'}\right)^{2d+1-n}, \qquad V' = V\left(\frac{k}{OP'}\right)^{n-1}.$$

When the law of force is the inverse distance $n = 1$, and the potential of the attracting mass takes a different form. In this case the quantity here called V, after multiplication by $n-1$, becomes the *mass* of the body and therefore the theorems of inversion though they no longer apply to the attractions of bodies will still enable us to find their masses when their densities vary as some power of the distance from a point. See *Quarterly J.*, May 1857.

Ex. 2. The potential of a homogeneous spherical surface at a point P is $4\pi a\rho$ or $4\pi a^2\rho/CP$ according as P is inside or outside the surface, where C is the centre and a is equal to the radius. It is required to invert this theorem with regard to an external point O.

Since the product of the segments $OQ \, . \, OQ'$ is constant in a sphere, it is clear that if we take k equal to the length of the tangent OT, the sphere will be its own inverse. When only one sphere occurs in the system this choice of the value of k will simplify the process, but when there are several spheres it will be more convenient to keep the value of k indeterminate.



If P is within the sphere, the inverse point P' is also within the sphere. By (4) the density of the inverse sphere at Q' is equal to $\rho\,(k/OQ')^3$, and its potential at P' is $4\pi a\rho k/OP'$.

If P is without the sphere, P' is also without. The density at Q' of the inverse system is the same as before, but the potential at P' is $\dfrac{4\pi a^2\rho}{CP}\cdot\dfrac{k}{OP'}$. Let C' be a point on the straight line OC such that C and C' are inverse points. Then by the similar triangles COP, $C'OP'$ we have $CP\,.\,OP'=OC\,.\,C'P'$. The potential at P' is therefore $\dfrac{4\pi a^2\rho}{OC}\cdot\dfrac{k}{C'P'}$.

If M' is the mass of the inverse system the relation between M' and ρ may be easily deduced from either of these expressions for the potential. Taking the first, where P' is inside the sphere, we notice that since every element of the sphere is equally distant from the centre, the potential at the centre is M'/a. Hence putting P' at the centre and comparing the two values of the potential, we have $M'=4\pi\rho a^2 k/OC$. Taking the second case, when P' is without the sphere, we notice that the potential at a very distant point must be mass divided by distance. Hence equating these two values of the potential, we have the same result as before.

Taking both these results, we arrive at the following inverse theorem.

Let a mass M' be distributed over a spherical surface, centre C, so that its density at any point Q' is $\rho\,(k/OQ')^3$, where O is an external point, and k is the length of the tangent from O. Then $\rho=M'c/4\pi a^2 k$, where $c=OC$, and the potential at any point P' is $M'\dfrac{c}{a}\dfrac{1}{OP'}$ or $\dfrac{M'}{C'P'}$, according as P' lies within or without the sphere. The points C' and C are inverse points with regard to O, and it is easy to see that C' lies on the polar line of O.

The potential of this heterogeneous spherical stratum at all external points is the same as if its whole mass M' were collected at C', and at all internal points is the same as if a mass $M'c/a$ were collected at O.

It has been shown in Art. 109 that the attraction of a finite body at a very distant point is the same as if the whole mass were collected into its centre of gravity. It follows that the point C' is the centre of gravity of the heterogeneous spherical stratum.

Ex. 3. If the density of a spherical surface vary as the inverse cube of its distance from an internal point O, find its potential at any point.

If the centre of inversion O is inside the primitive sphere we can still make the sphere its own inverse by drawing OQ' from O in the direction opposite to OQ and taking k^2 equal to the product of the segments of all chords through O. With these changes we may show that the potential at all external points is the same as if its whole mass M' were collected at O and at all internal points is the same as if the mass $M'c/a$ were collected at C'.

Ex. 4. The potential of a homogeneous solid sphere at an external point P is $\frac{4}{3}\pi\rho a^3/CP$, where C is the centre and a the radius. Invert this theorem with regard to an external point O.

The result is that the potential at an external point of a heterogeneous sphere whose density at any point Q' is $\rho\,(k/OQ')^5$ is the same as if its whole mass M' were collected into a fixed point C'. This point C' is the inverse of the centre with regard to O and is also the centre of gravity of the sphere. The constant ρ may be found from the relation $M'c=\frac{4}{3}\pi\rho a^3 k$, where $c=OC$, and k is the length of the tangent from O.

Ex. 5. A heterogeneous spherical shell is bounded by eccentric spheres whose

radii are a, b, and its density at any point Q is m/OQ^5, where m is a constant and O a given external point. Show that its potential at any point P is

$$\tfrac{2}{3}\pi m\left[3\left(\frac{a^2}{f^4}-\frac{b^2}{g^4}\right)\frac{1}{OP}-\left(\frac{AP^2}{OA^2}-\frac{BP^2}{OB^2}\right)\frac{1}{OP^3}\right],$$

where A and B are points where the polar lines of O intersect the diameters drawn through O and f, g are the tangents from O.

Ex. 6. An infinitely thin layer of matter is placed on the surface of elasticity $c^2r^4=a^2x^2+b^2y^2+c^2z^2$, so that the surface density at any point distant r from the centre varies as p/r^5, where p is the perpendicular from the origin on the tangent plane. Show that the potential at any external point is the same as if the whole mass were collected at its centre of gravity.

152. **Inversion from a line.** Instead of inverting the attracting system with regard to a point O we may invert it with regard to some straight line Oz. Let a point Q move in any manner, and let QN be a perpendicular on the axis Oz. If on NQ we take a point Q' so that $NQ \cdot NQ'=k^2$, where k is a given constant, then Q' is the inverse of Q with regard to the axis of z.

With this definition it is clear that any cylindrical surface with its generators parallel to Oz inverts into another cylindrical surface also having its generators parallel to that axis. This method of inversion will therefore help us to deduce the potential of one cylindrical surface or solid from that of its inverse. We shall suppose that the density of the cylindrical body is uniform along any generating line but varies from one generator to another.

153. The attraction of an infinite rod parallel to the axis of z at any point P on the plane of xy is known to be $2m/QP$, where Q is the intersection of the rod with the plane of xy and m is the line density. The potential of such a rod at P is therefore $V=C-2m\log QP$, where C is some constant, Art. 42. Let us invert this rod with regard to the axis of z into a parallel rod and P into another point P'. Supposing the inverse rod to have the same line density as the primitive rod, its potential at P' is $V'=C-2m\log Q'P'$. But by Art. 149 $P'Q'=PQ\cdot\dfrac{OP'}{OQ}$. Hence

$$V'+2m\log OP'=V+2m\log OQ \qquad\qquad (1).$$

Let there be a system of rods intersecting the plane of xy in the points Q_1, Q_2, &c., and let the inverse rods intersect the same plane in Q_1', Q_2', &c. Let m_1, m_2, &c. be the line densities of the several pairs. Then for each pair we have an equation similar to (1); adding all these together we find

(Potential at P' of inverse system)

– (Potential at P' of the whole mass collected at the axis)

= (Potential at P of given system) – (Potential at O of given system).

154. If the primitive system of rods intersect the plane of xy in an arc or an area, the inverse system will also be arranged in the same way. To compare the densities we observe that the masses of the given system and the inverse are the same but differently distributed. If the locus of Q is an arc, the ratio of the elementary arcs at Q', Q is equal to OQ'/OQ, and the ratio of the line densities is therefore equal to OQ/OQ', i.e. $(k/OQ')^2$. If the locus of Q is an area, the ratio of the surface densities is equal to $(k/OQ')^4$.

We should notice that m is the mass per unit of length of a rod. Hence when the attracting rods form a cylindrical surface whose surface density is ρ, we have $m=\rho ds$, where ds is an element of arc of the section of the cylinder by a plane perpendicular to the axis. For example, in the case of a right circular cylinder of

radius a we have $\Sigma m = 2\pi a\rho$. If the rods form a cylindrical volume of density ρ, we have $m = \rho dA$ where dA is an element of area of the curve of section.

Ex. 1. A heterogeneous stratum is placed on a right circular cylinder, the density being uniform along any generator. It is required to compare the potentials at an internal and an external inverse point.

If we invert the system with regard to the axis and the radius k of inversion be the radius of the cylinder, the stratum inverts into itself. If P, P' be the internal and external points, V, V' the potentials, we have by Art. 153

$$V' - (C' - 2\Sigma m \log OP') = V - V_0.$$

Collecting all the constant terms into one, we have

$$V' - V = A - 2\Sigma m \log OP'.$$

The corresponding proposition for a sphere is given in Art. 72.

Ex. 2. Invert the following theorem with regard to an eccentric internal straight line. The potential of a homogeneous right circular cylindrical surface at any internal point is constant and equal to that along the axis.

The resulting theorem is as follows. If matter be distributed in a thin stratum over a right circular cylinder so that the surface density at any point Q' is proportional to the inverse square of the distance of Q' from an internal straight line OZ parallel to the generators, the potential at any external point is the same as if the whole mass were evenly distributed over the straight line OZ.

155. Let Q_1, $Q_2, \dots Q_n$, be n points arranged at equal distances on the circumference of a circle of radius ρ. Taking the centre O as origin, let the polar coordinates of these points be (ρ, ϕ), $(\rho, \phi + a)$, $(\rho, \phi + 2a)$ &c. where $na = 2\pi$. Let P be any point and let (r, θ) be its coordinates. By De Moivre's property of the circle we have

$$r^{2n} - 2r^n\rho^n \cos n(\theta - \phi) + \rho^{2n} = PQ_1^2 \,.\, PQ_2^2 \dots PQ_n^2 \quad\text{.................(1).}$$

Let us now take two other points Q', P' whose coordinates (ρ', ϕ') and (r', θ') are such that $\rho' = c\left(\frac{\rho}{c}\right)^n$, $\phi' = n\phi$; $r' = c\left(\frac{r}{c}\right)^n$, $\theta' = n\theta$ where c is any constant. It immediately follows that the left side of (1) is equal to $c^{2(n-1)} \,.\, (P'Q')^2$. Taking the logarithm of both sides, we find

$$\log P'Q' + (n-1)\log c = \log PQ_1 + \log PQ_2 + \text{\&c.} + \log PQ_n \quad\text{...........(2).}$$

Let us now suppose that two infinite thin rods, each of uniform line density m, are placed perpendicularly to the plane of the circle at P and P' respectively. It follows at once from equation (2) that the potential of the second rod at Q' differs by a constant from the sum of the potentials of the first rod at the points Q_1, $Q_2, \dots$ Q_n. See Art. 153.

In the same way, by properly placing pairs of corresponding rods we may build up two corresponding cylindrical bodies, which have the property that the potential of the second body at Q' differs by a constant from the sum of the potentials of the first at $Q_1 \dots Q_n$.

We may express this result in the form of a theorem. *An infinitely long cylindrical body has its density uniform along any generating line and attracts according to the law of nature. The body, being referred to cylindrical coordinates with the axis of z parallel to the generators, is transformed into a second cylindrical body by moving each cylindrical element* (r, θ) *into the position* $(r', \theta',)$ *where* $r' = c\left(\frac{r}{c}\right)^n$, $\theta' = n\theta$ *without altering the mass of element. If the potentials of the original body at the n points* (ρ, ϕ), $(\rho, \phi + a)$, $(\rho, \phi + 2a)$ *&c. be* V_1, V_2, V_3 *&c. then the potential of the transformed body at* (ρ', ϕ'), *where* $\rho' = c\left(\frac{\rho}{c}\right)^n$, $\phi' = n\phi$, *differs by a constant from the sum* $V_1 + V_2 + \text{\&c.} + V_n$.

If the first body be a continuous cylindrical solid, the second body may be made also continuous by altering the areas of the sections of the transformed elements, keeping the mass unchanged. Since the elementary areas at P, P' are respectively $rd\theta dr$ and $r'd\theta'dr'$ we easily see that the volume densities at P, P' must be in the ratio of $(nr')^2$ to r^2.

If the first body be a continuous surface, the second body may be made also a continuous surface. Since the elementary arcs in which the surfaces cut the plane of xy are as $r^2d\theta/p$ to $r'^2d\theta'/p'$, where p and p' are the perpendiculars from O on the tangent planes, the densities per unit of arc must be as p/r^2 to p'/nr'^2.

Ex. Thin layers of attracting matter are placed on the cylinders

$$A(x^4+y^4)+2Bx^2y^2=1$$

$$Ax^6+3(3B-2A)x^4y^2+3(3A-2B)x^2y^4+By^6=1;$$

if the surface densities are proportional respectively to r^2p and r^4p, where r is the distance from the axis and p is the perpendicular on the tangent plane, prove that the potentials are constant at all internal points.

By Art. 56, the attraction of a thin shell bounded by similar cylinders properly placed is zero at any internal point. Applying the preceding theorem we deduce the surface densities.

Application of Laplace's functions to spheres and solids of revolution.

156. In many parts of the theory of Attractions, the integrations are shortened and made more comprehensive by the use of Laplace's functions. In other parts the necessary processes could not be effected without their help. There are several treatises on these functions from which the reader may acquire a knowledge of this important branch of Pure Mathematics. The propositions however which are wanted in Attractions are not very numerous and these books contain much more than is here required. At the same time the subject of Attractions is generally approached by the student at a period of his course when he has not yet reached the proper study of these functions. For these reasons it seems proper to make a preliminary statement of a few elementary theorems which the reader acquainted with Laplace's functions may pass over. These are given in the small print in the following articles.

157. **Expansion of the inverse distance.** Let P, P' be two points, one of which will afterwards be taken as a point of the attracting mass and the other as the point at which the attraction is required. Let (x, y, z), (x', y', z') be their Cartesian coordinates referred to any rectangular axes, (r, θ, ϕ), (r', θ', ϕ') their corresponding polar coordinates. Let R be the distance between the points and let $p=\cos POP'$. We therefore have

$$\frac{1}{R}=\frac{1}{\sqrt{\{(x-x')^2+(y-y')^2+(z-z')^2\}}}=\frac{1}{\sqrt{\{r^2-2rr'p+r'^2\}}}\;\ldots\ldots\ldots\ldots(1).$$

It will be found convenient to expand $1/R$ in a convergent series of ascending powers of either r/r' or r'/r. Supposing first $r<r'$, we write $h=r/r'$. We then have by the binomial theorem

$$(1-2ph+h^2)^{-\frac{1}{2}}=1+\tfrac{1}{2}(2ph-h^2)+\tfrac{3}{8}(2ph-h^2)^2+\ldots$$

Expanding these terms and writing P_1, P_2, &c. for the coefficients of the several powers of h

$$(1-2ph+h^2)^{-\frac{1}{2}}=1+P_1h+P_2h^2+\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(2).$$

The terms containing h^n are evidently the first in $(2ph-h^2)^n$, the second in $(2ph-h^2)^{n-1}$, and so on. It is therefore clear that P_n *is a rational integral function of* p, *whose highest power is* p^n *and whose powers descend two at a time.* Thus P_n is of the form

$$P_n = A_0 p^n + A_2 p^{n-2} + \dots\dots (3),$$

where A_0, A_2 &c. are constants.

These constants are easily found when n is a small integer by the use of the binomial theorem in the manner shown above, thus

$$P_1 = p, \quad P_2 = \tfrac{1}{2}(3p^2-1), \quad P_3 = \tfrac{1}{2}(5p^3-3p) \text{ \&c.}$$

When n is too large an integer to apply this method we may use some one of the many rules which have been constructed for this purpose. For these however we must refer the reader to treatises on the subject of these functions. The most useful of these is perhaps that given by Rodrigues, viz.

$$P_n = \frac{1}{2^n \underline{|n}} \frac{d^n}{dp^n}(p^2-1)^n.$$

The two following theorems are also useful.

$$nP_n - (2n-1)pP_{n-1} + (n-1)P_{n-2} = 0.$$

$$\frac{dP_n}{dp} = (n+1)\frac{P_{n+1} - pP_n}{p^2-1}.$$

By using the first of these we may find any of the coefficients P_0, P_1, P_2, &c. when the two preceding are known.

158. It will be useful to notice that when $p=1$ the equation (2) reduces to

$$(1-h)^{-1} = 1 + P_1 h + P_2 h^2 + \dots.$$

Expanding the left-hand side by the binomial theorem, we see that each of the functions P_1, P_2, &c. is equal to unity when $p=1$.

159. The function P_n is usually called a *Legendre's function* of the nth order. It is sometimes written in the form $P_n(p)$ when it is desired to call attention to the independent variable. Regarding one of the two radii vectores OP, OP' as a fixed axis and the other as capable of moving into all positions round the origin, P_n is a function of the inclination of the latter to the fixed axis. The fixed radius vector is there called the axis of reference of the function or more shortly the axis of the function.

160. If $(\alpha', \beta', \gamma')$ are the direction cosines of OP', we have by projecting OP on OP'

$$pr = \alpha' x + \beta' y + \gamma' z,$$

$$\therefore \quad P_n r^n = A_0(\alpha' x + \beta' y + \gamma' z)^n + A_2(\alpha' x + \beta' y + \gamma' z)^{n-2}(x^2+y^2+z^2) + \dots.$$

Regarding OP' as fixed in space and OP as moving about O we see that $P_n r^n$ *is a homogeneous rational and integral function of the coordinates of* P.

The quantity $1/R$, regarded as a function of the variables (x, y, z), is known to satisfy Laplace's equation, Art. 76. Since this is true whatever (x', y', z') may be, provided they are fixed, it follows that the coefficient of every power of $1/r'$ in the expansion

$$\frac{1}{R} = \frac{1}{r'} + \frac{P_1 r}{r'^2} + \frac{P_2 r^2}{r'^3} + \dots\dots (4).$$

satisfies Laplace's equation.

161. Any homogeneous function of (x, y, z) which satisfies Laplace's equation has been called by Thomson a *spherical harmonic function*. Its degree may be any positive or negative integer, it may be fractional or imaginary.

When the function is such that it may be written in the form $r^n f(\theta)$ where θ is

the inclination of the radius vector to a fixed straight line, it is called sometimes a *zonal* and sometimes an *axial* spherical harmonic.

We therefore see that $P_n r^n$ is an axial spherical harmonic of the nth order.

162. The expansion (4) has been made in powers of r/r' on the supposition that r is less than r'. If the contrary be the case we must make the expansion in powers of r'/r in order that the series may be convergent. We then have

$$\frac{1}{R}=\frac{1}{r}+\frac{P_1 r'}{r^2}+\frac{P_2 r'^2}{r^3}+\dots\dots\dots\dots(5).$$

It follows in the same way that the coefficient of r'^n viz. $P_n r^{-(n+1)}$ is a homogeneous function of the $-(n+1)$th order which satisfies Laplace's equation. Thus both $P_n r^n$ and $P_n r^{-(n+1)}$ are axial harmonics of different orders.

163. **Potential of a body.** To apply these expansions to find the potential of a body, we regard (x', y', z') as the coordinates of any particle m of the attracting mass. We now multiply $1/R$ by m and sum or integrate the result for all the attracting particles. At some points of the body we may have $r' > r$, at others $r > r'$; we therefore may have to use both the expansions (3) and (4) each for the appropriate portion of the attracting mass. In this way we find

$$V=\Sigma\frac{m}{R}=Y_0+Y_1 r+Y_2 r^2+\dots+\frac{Z_0}{r}+\frac{Z_1}{r^2}+\frac{Z_2}{r^3}+\dots\dots\dots(6),$$

where $$Y_n=\Sigma\frac{mP_n}{r'^{n+1}} \text{ and } Z_n=\Sigma m r'^n P_n.$$

These summations cannot be effected until the form and law of density of the heterogeneous body are known. We notice however that both Y_n and Z_n are the sums of a number of Legendre's functions with coefficients and axes depending on the given structure and shape of the body. Regarded therefore as a function of (x, y, z) both $Y_n r^n$ and $Z_n r^n$ are integral rational spherical harmonics.

164. **Laplace's equations.** In this way we have been led to an expansion of V in powers of r which must hold for all attracting masses. Let this be written $V=\Sigma Y_n r^n$, where n may be either a positive or a negative integer. Substituting this series for V in Laplace's equation as expressed in polar coordinates (Art. 78) and equating the coefficient of r^n to zero, we have

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dY_n}{d\mu}\right\}+\frac{1}{1-\mu^2}\frac{d^2 Y_n}{d\phi^2}+n(n+1)Y_n=0\dots\dots\dots(7),$$

where $\mu=\cos\theta$.

The corresponding equation for Y_m is found by writing m for n. If we choose m so that $m(m+1)=n(n+1)$ we have $m=n$ or $m=-(n+1)$. It follows that there are two powers of r, and only two, viz. r^n and $r^{-(n+1)}$ such that their coefficients in the series (6) viz. Y_n and Z_n, satisfy the differential equation (7). It appears therefore that Y_n and Z_n are both solutions of the differential equation (7) and differ only in the arbitrary functions or constants which occur in the solution.

Any function of two independent angular coordinates (such as the direction angles θ, ϕ of the radius vector) which satisfies equation (7) is called a *Laplace's function*. Thus Y_n is a Laplace's function of the order n. The corresponding function $Y_n r^n$ when expressed in terms of (x, y, z) satisfies Laplace's equation and is a spherical harmonic, Art. 161. A Laplace's function when expressed as a function of the Cartesian coordinates of the point at which the radius vector intersects some given sphere with its centre at the origin is called a *spherical surface harmonic*.

165. If θ', ϕ' be the direction angles of a fixed radius vector OP' and $\cos POP' = p$, we have $p = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$.

The Legendre's function P_n is therefore a symmetrical function of θ, ϕ and θ', ϕ'. Regarded as a function of θ, ϕ, we see, by comparing the series (4) and (5) with (6), that P_n is a special case of Y_n. It follows that P_n must also satisfy the Laplacian equation (7).

If the axis of the function P_n, i.e. OP', be taken as the axis of reference, we have $\mu = p$ and $dP_n/d\phi = 0$. The function P_n must therefore satisfy the differential equation

$$\frac{d}{dp}\left\{(1-p^2)\frac{dP_n}{dp}\right\} + n(n+1)P_n = 0 \quad\text{.......................} \quad (8).$$

The general solution of the differential equation (8) has two arbitrary constants. To find the general solution when a partial solution has been found we use a rule given in the theory of differential equations (see Forsyth's *Diff. Eq.* Art. 58). The general solution is thus found to be

$$AP_n + BP_n \int \frac{dp}{P_n^2(p^2-1)},$$

where A and B are the two arbitrary constants. Since P_n is an integral rational function of p we may by using partial fractions effect this integration. The process is rather long and the results will not be required. It will be sufficient to notice that the part of the solution derived from the integral is not an integral rational function of p. It follows that the only integral rational solution is AP_n.

166. We have seen in Art. 163 that the potential of any body can be expanded in a series of spherical harmonics of integral orders. In this expansion $Y_n r^n$ and $Z_n r^n$ are both integral and rational functions of x, y, z of a positive integral order. Changing to polar coordinates we find that Y_n is an integral function of $\cos\theta$, $\sin\theta\cos\phi$, $\sin\theta\sin\phi$. Expanding the powers of $\sin\phi$, $\cos\phi$ in multiple angles, we have

$$Y_n = A_0 + (A_1\cos\phi + B_1\sin\phi) + (A_2\cos 2\phi + B_2\sin 2\phi) + \ldots + (A_n\cos n\phi + B_n\sin n\phi) \ldots (9),$$

where A_0, $A_1 \ldots A_n$, $B_1 \ldots B_n$ are all integral and rational functions of $\sin\theta$ and $\cos\theta$.

Substituting this value of Y_n in (7), we see that both A_k and B_k satisfy

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dA_k}{d\mu}\right\} + n(n+1)A_k = \frac{k^2}{1-\mu^2}A_k \quad\text{....................}\quad (10),$$

where $\mu = \cos\theta$.

Since the equation (10) reduces to the form (8) when $k=0$, we have $A_0 = a_0 P_n(\mu)$, where a_0 is an arbitrary constant.

The values of A_1, B_1 &c. will not be required; it will therefore be sufficient to mention that their values found from equation (10) are

$$A_k = a_k(\sin\theta)^k\frac{d^kP_n}{d\mu^k}, \qquad B_k = b_k(\sin\theta)^k\frac{d^kP_n}{d\mu^k}.$$

167. **Three theorems.** The great utility of Laplace's functions depends on three theorems to which we shall now turn our attention.

Theorem I. If Y_m, Y_n be two Laplace's functions of different orders then $\int Y_m Y_n d\omega = 0$, where $d\omega$ is an elementary solid angle and the integration extends over the whole surface of the unit sphere.

The proof which we shall adopt here is that by which Sir W. Thomson deduces Laplace's result from Green's theorem. Putting $V = Y_m r^m$, $V' = Y_n r^n$, we have by applying Green's theorem to the surface of a sphere of radius a whose centre is at the origin
$$\int V \frac{dV'}{dr} d\sigma = \int V' \frac{dV}{dr} d\sigma.$$

Substituting for V, V' we have $\quad a^{m+n+1} n \int Y_m Y_n d\omega = a^{m+n+1} m \int Y_m Y_n d\omega$;

hence unless m and n are equal, we have $\quad \int Y_m Y_n d\omega = 0$.

When m and n are positive these values of V and V' are both finite throughout the sphere. If however m or n is negative it is necessary to integrate over the two surfaces of a spherical shell to avoid the infinity at the centre. If a and b be the radii we then have
$$(a^{m+n+1} - b^{m+n+1})\, n \int Y_m Y_n d\omega = (a^{m+n+1} - b^{m+n+1})\, m \int Y_m Y_n d\omega.$$

It follows that $\int Y_m Y_n d\omega = 0$ unless $m = n$ or $m + n + 1 = 0$.

Since the Legendre's function P_n is merely a special case of Y_n, viz. that in which the function is symmetrical about the axis of reference, we have in the same way
$$\int P_m P_n d\omega = 0, \qquad \therefore \int_{-1}^{1} P_m P_n dp = 0 \quad \text{.................} \quad (12).$$

The latter result is obtained from the former by writing $d\omega = \sin\theta\, d\theta\, d\phi$, and integrating with regard to ϕ from 0 to 2π.

168. **Theorem II.** Let Y_n be a Laplace's function of the angular coordinates (θ, ϕ) and P_n a Legendre's function of the same coordinates having (θ', ϕ') for its axis. Let both these be of the same order viz. n, then
$$\int Y_n P_n d\omega = \frac{4\pi}{2n+1} Y_n' \quad \text{..............................} (13),$$
where the integration extends over the whole unit sphere, and Y_n' is the value of Y_n when (θ', ϕ') have been substituted for (θ, ϕ).

We shall first prove that this theorem is true for the particular case $\int P_n^2 d\omega$ and thence by using the expression for Y_n given in Art. 166 deduce its truth in the more general case in which $r^n Y_n$ is an integral rational spherical harmonic.

We have by (2) if $h < 1$, Art. 157 $\qquad (1 - 2ph + h^2)^{-\frac{1}{2}} = 1 + P_1 h + P_2 h^2 + \dots$.

Squaring both sides and integrating between the limits -1 and $+1$,
$$\int (1 - 2ph + h^2)^{-1} dp = \int (1 + P_1 h + P_2 h^2 + \dots)^2 dp.$$

The left-hand side, by an easy integration, becomes
$$\frac{1}{h} \{\log(1+h) - \log(1-h)\} = 2\{1 + \tfrac{1}{3}h^2 + \tfrac{1}{5}h^4 + \dots\}.$$

Since $\int P_m P_n dp = 0$ when m and n are unequal, the right-hand side becomes
$$\int (1 + P_1^2 h^2 + P_2^2 h^4 + \dots)\, dp.$$

Equating the coefficients of h^{2n}, we find
$$\int P_n^2 dp = \frac{2}{2n+1} \quad \text{....................................} (14),$$
where the limits of integration are $p = -1$ to $+1$.

To find the value of $\int Y_n P_n d\omega$, let us take as the axis of z, the axis of P_n, so that $P_n = P_n(\mu)$, where $\mu = \cos\theta$. Also $d\omega = \sin\theta d\theta d\phi$ becomes $-d\mu d\phi$. The limits of integration are $\mu = 1$ to -1, $\phi = 0$ to 2π.

Taking the value of Y_n given in Art. 166 viz.

$$Y_n = a_0 P_n(\mu) + \Sigma (A_k \cos k\phi + B_k \sin k\phi),$$

we notice that $\int \cos k\phi d\phi = 0$ and $\int \sin k\phi d\phi = 0$ when the limits of ϕ are 0 to 2π. Hence

$$\int Y_n P_n d\omega = -a_0 \iint P_n^2 d\mu d\phi = a_0 \cdot 2\pi \cdot \frac{2}{2n+1}.$$

It remains to find the value of a_0. Referring to equation (10) of Art. 166, we see $A_k = 0$ and $B_k = 0$ when $\mu = 1$ except when $k = 0$. Also $P_n(\mu) = 1$ when $\mu = 1$. Thus a_0 is the value of Y_n at the point where the positive direction of the axis of z cuts the unit sphere. Since the axis of P_n has been taken as the axis of z it follows that a_0 is the value of Y_n at the positive extremity or pole of the axis of P_n, and this value has been represented in the enunciation by Y_n'.

169. **Theorem III.** Any function of the two angular coordinates of the radius vector can be expanded in a series of Laplace's functions, and the expansion can be made in only one way.

For a discussion of this important theorem we must refer the reader to the treatises on these functions. It will suffice here if we consider how we may practically use the theorem in those simpler cases which generally occur in the theory of attraction.

Let us first suppose that the given function is an integral rational function of the direction cosines of the radius vector, i.e. of $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, and $\cos\theta$. On transforming to Cartesian coordinates and multiplying each term by the proper power of r the function becomes an integral rational function of x, y, z, which we can arrange in a series of homogeneous functions. Taking any one of these, say $f_n(x, y, z)$, we shall show how it may be expanded in a series of spherical harmonics combined with powers of r. Thence (if it be necessary) we deduce the expansion in Laplace's functions by giving r any constant value.

Subtract from f_n the expression $(x^2+y^2+z^2)f_{n-2}$, where f_{n-2} is an arbitrary integral and rational function of (x, y, z) of the $(n-2)$th degree, viz.

$$f_{n-2} = A_0 x^{n-2} + A_1 x^{n-3} y + B_1 x^{n-3} z + \ldots.$$

Substituting $V = f_n - (x^2+y^2+z^2) f_{n-2}$ in $\nabla^2 V$, there results a homogeneous function of (x, y, z) of the $(n-2)$th degree, which therefore contains as many terms as there are ways of making homogeneous products of x, y, z of that degree. But f_{n-2} is an arbitrary homogeneous function of the same degree and contains an equal number of terms. There are therefore just enough arbitrary constants A_0, A_1, B_1 &c. to enable us to make the coefficients of every term in $\nabla^2 V$ equal to zero. Assuming that the linear equations thus formed to find A_0, A_1 &c. are not inconsistent with each other, the expression $f_n - (x^2+y^2+z^2) f_{n-2} = S_n$ satisfies Laplace's equation and is therefore a spherical harmonic.

Repeating this process with the function f_{n-2}, we have

$$f_{n-2} - (x^2+y^2+z^2) f_{n-4} = S_{n-2},$$

and so on. We finally end with a constant or an expression of the first degree according as n is an even or odd integer.

Writing r^2 for $x^2+y^2+z^2$ we have $f_n = S_n + r^2 S_{n-2} + r^4 S_{n-4} + \ldots$, where S_n, S_{n-2} &c. are all spherical harmonics. It should be noticed that this equality is a mere algebraical transformation, and involves no assumptions as to the meaning of the letters.

If we now regard r as the radius of the unit sphere or any suitable sphere, S_n, S_{n-2} &c. become Laplace's functions, and the required expansion has been made.

When the function does not contain powers of x, y, z above the cube, this

process will be unnecessary, for the arrangement in harmonics can then be generally performed at sight.

170. When the Cartesian equivalent of the given function is not an integral rational function of the coordinates, an expansion in a finite number of terms cannot be obtained. We then proceed in another way. Assume that the expansion can be effected in a convergent series, say $f(\theta, \phi) = Y_0 + Y_1 + Y_2 + \ldots$, where Y_n is a Laplace's function of the nth order. Let P_n be the Legendre's function having (θ', ϕ') for its axis, so that P_n is a symmetrical function of (θ, ϕ) and (θ', ϕ'); Art. 165. Multiply both sides of the equation by P_n and integrate over the whole surface of the unit sphere; then by Art. 168 $\iint f(\theta, \phi)\, P_n d\mu d\phi = \frac{4\pi}{2n+1} Y_n'$,

where Y_n' is the value of Y_n when (θ', ϕ') have been written for (θ, ϕ). When the integration on the left-hand side has been effected, the result will be a known function of θ', ϕ' only. Since θ', ϕ' are arbitrary we can replace them by θ, ϕ and thus the form of Y_n has been found.

Laplace's expansion is an extension to two independent variables of Fourier's expansion of a function of one variable in a series of sines and cosines of its multiples, and like that theorem is subject to limitations. The process of expansion given above is not in any way a proof, it is to be regarded as merely a convenient method of applying Laplace's theorem to special cases. It fails to give the limitation and must be used with caution when the function to be expanded is not single valued.

171. Ex. 1. What are the conditions that

$$(1)\quad ax + by + cz, \qquad (2)\quad Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy$$

may be spherical harmonics? The first is always so, the second when $A + B + C = 0$.

Ex. 2. Expand $\sin^3\theta \cos^3\phi$ in Laplace's function.

This is the same as p^3 if the axis of x be taken as the axis of reference. Now $P_3 = \frac{1}{2}(5p^3 - 3p)$, hence $p^3 - \frac{2}{5}P_3 = \frac{3}{5}p$. The result is $p^3 = \frac{2}{5}P_3 + \frac{3}{5}P_1$.

Ex. 3. Expand $\sin^2\theta \sin\phi \cos\phi + \cos^3\theta$ in Laplace's functions.

The result is $Y_1 + Y_2 + Y_3$, where $Y_1 = \frac{3}{5}\cos\theta$, $Y_2 = \sin^2\theta \sin\phi\cos\phi$,

$$Y_3 = \tfrac{1}{5}(5\cos^3\theta - 3\cos\theta).$$

Ex. 4. Expand $\log(1 + \operatorname{cosec}\frac{1}{2}\theta)$ in Legendre's function. [Coll. Ex.]

The result is $P_0 + \frac{1}{2}P_1 + \frac{1}{3}P_2 + \frac{1}{4}P_3 + \ldots$.

Ex. 5. Prove the equalities

$$P_0^2 + 3P_1^2 + \&c. + (2n+1)P_n^2 = (n+1)^2 P_n^2 - (p^2-1)\left(\frac{dP_n}{dp}\right)^2,$$

$$\left(\frac{dP_0}{dp}\right)^2 + 3\left(\frac{dP_1}{dp}\right)^2 + \&c. + (2n+1)\left(\frac{dP_n}{dp}\right)^2 = \frac{1}{3}\left\{(n+2)^2\left(\frac{dP_n}{dp}\right)^2 - (p^2-1)\left(\frac{d^2P_n}{dp^2}\right)^2\right\}.$$

172. **Attraction of a stratum.** *A thin heterogeneous stratum of attracting matter is placed on a sphere of radius a. It is required to find its potential at any internal or external point.*

Let ρ be the surface density at any point Q of the sphere, $d\sigma$ an element of area at Q; θ, ϕ the polar coordinates of Q, then $d\sigma = \sin\theta d\theta d\phi$. Let P be the point at which the attraction is required, and let the coordinates of P be (r', θ', ϕ').

If R be the distance between the points Q and P, the potential

of the whole stratum at P is $V=\int \frac{\rho d\sigma}{R}$. Let p be the cosine of the angle between the positive directions of the radii vectores OQ and OP, then $R^2=a^2+r'^2-2apr'$.

If the point P is inside the sphere, r' is less than a, and we may expand $1/R$ in a convergent series of ascending powers of r'/a. If the point attracted is outside the sphere, we must expand in powers of a/r'. Since R is a symmetrical function of a and r we have

$$\frac{1}{R}=\frac{1}{a}\left\{P_0+P_1\frac{r'}{a}+P_2\left(\frac{r'}{a}\right)^2+\ldots\right\}$$

or

$$=\frac{1}{r'}\left\{P_0+P_1\frac{a}{r'}+P_2\left(\frac{a}{r'}\right)^2+\ldots\right\}.$$

The density ρ is a given function of the coordinates of Q; let it be expanded in a series of Laplace's functions or surface harmonics, thus $\rho=Y_0+Y_1+Y_2+$...........

Substituting these values of ρ and $1/R$ in the expression for V, we have by the theorems I. and II. in Arts. 167, 168,

$$V=4\pi a\left\{Y_0'+\frac{1}{3}Y_1'\frac{r'}{a}+\frac{1}{5}Y_2'\left(\frac{r'}{a}\right)^2+\ldots\right\}.$$

$$V'=\frac{4\pi a^2}{r'}\left\{Y_0'+\frac{1}{3}Y_1'\frac{a}{r'}+\frac{1}{5}Y_2'\left(\frac{a}{r'}\right)^2+\ldots\right\}.$$

according as r' is less or greater than a. The first of these two expansions gives the potential at any internal point, the second at any external point.

If Y_n is expressed as a function of the angular coordinates (θ', ϕ') of Q, then as already explained (Art. 168) Y_n' is the value of Y_n when the polar coordinates of the attracted point P have been written for (θ, ϕ). If however Y_n is expressed as a homogeneous function of the Cartesian coordinates (x, y, z) of Q, then Y_n' is obtained from Y_n by writing the Cartesian coordinates of P for (x, y, z) and multiplying the result by $(a/r')^n$.

173. Ex. 1. The surface density at any point Q of a sphere is a quadratic function of the Cartesian coordinates of Q. Find the potential at any point whose coordinates are (x', y', z').

Let the surface density ρ be given by $\rho=Ax^2+By^2+Cz^2+2Dyz+2Ezx+2Fxy$. Let us represent this function by $f(x, y, z)$.

As this function would be a spherical harmonic if $A+B+C=0$, we make the necessary expansion in surface harmonics by subtracting and adding $G(x^2+y^2+z^2)$, where $3G=A+B+C$. We therefore have $\rho=Y_0+Y_2$, where

$$Y_0=Ga^2,\quad Y_2=f(x, y, z)-G(x^2+y^2+z^2).$$

The required potential at the point P is therefore

$$V = 4\pi a \left\{ Y_0' + \tfrac{1}{5} Y_2' \left(\frac{r'}{a}\right)^2 \right\},$$

$$\text{or} \quad V' = \frac{4\pi a^2}{r'} \left\{ Y_0' + \tfrac{1}{5} Y_2' \left(\frac{a}{r'}\right)^2 \right\},$$

according as P is inside or outside the sphere. Here $Y_0' = Ga^2$ and

$$Y_2' = \{f(x', y', z') - Gr'^2\} \left(\frac{a}{r'}\right)^2.$$

Substituting these values for Y_0' and Y_2' in the formulae for V and V' the required potentials have been found.

Ex. 2. The surface density at any point of a sphere is $\rho = mxy$: show that its potential at any point (x', y', z') is $\frac{4\pi am}{5} x'y'$ or $\frac{4\pi am}{5} x'y' \left(\frac{a}{r'}\right)^5$, according as the point is within or without the sphere.

Ex. 3. The surface density at any point of a sphere is $mxyz$: show that the potential at an internal point is $\frac{4}{7}\pi am\, x'y'z'$.

Ex. 4. If P, P' be two points on the same radius vector at distances r, r' from the centre such that $rr' = a^2$, prove that the potentials, V, V' at these points are connected by the equation $V' = Va/r'$. See Art. 72.

Ex. 5. If the surface density at any point Q be an integral rational function of the Cartesian coordinates of Q of a degree not higher than the nth, prove that the potential at any internal point P is an integral rational function of the Cartesian coordinates of P also of a degree not higher than the nth. What is the corresponding theorem for an external point?

174. **Attraction of a solid sphere.** *To find the potential of a solid heterogeneous shell bounded by concentric spheres when the density ρ at any point is a homogeneous function of the coordinates of the kth degree.*

Let the density ρ be expanded in a series of the form

$$\rho = r^k \{Y_0 + Y_1 + Y_2 + \dots\dots\dots\},$$

where Y_n is a Laplace's function of the angular coordinates. Let the radii of the shell be a and b.

Taking as an element any concentric shell bounded by the radii r and $r + dr$, its surface density is ρdr. Its potentials at an internal and external point are therefore respectively given by

$$V_s = 4\pi r^{k+1}\, dr \,\Sigma \frac{Y_n'}{2n+1} \left(\frac{r'}{r}\right)^n.$$

$$V_s' = \frac{4\pi r^{k+2}\, dr}{r'} \Sigma \frac{Y_n'}{2n+1} \left(\frac{r}{r'}\right)^n,$$

where the summation extends to all the values of n in the expansion of ρ. Integrating these expressions from $r = a$ to $r = b$ the required potentials follow at once.

175. Ex. 1. The density of a shell bounded by concentric spheres of radii a and b is given by $\rho = mxy$. Show that the potential at an internal point is $\frac{2}{5} m\pi (b^2 - a^2) x'y'$.

Ex. 2. The density of a solid sphere of radius a is given by $\rho = mxyz$. Show that its potential at an external point is $\frac{4}{63}\pi m a^9 \frac{x'y'z'}{r'^7}$.

176. **Nearly spherical bodies.** *The strata of equal density of a solid are nearly spherical and both its internal and external boundaries are surfaces of equal density. Find its potential at an internal and external point.*

Let any surface of equal density be $r = a + af(\theta, \phi, a)$, where a is a constant and f a function whose square can be neglected. The quantity a is the parameter of the strata, i.e. by its variation we pass from one stratum to another. Let the internal and external boundaries be defined by $a = a_0$ and $a = a_1$. Let the density of any stratum be $\rho = F(a)$.

Let the equation of the stratum be expanded in a series of Laplace's functions, viz. $r = a(1 + \Sigma Y_n)$ ....(1).

The solid bounded by this surface and any internal fixed concentric sphere of radius b may be regarded as a spherical shell of radius $a - b$, together with a stratum of surface density $a\Sigma Y_n$ placed on its external boundary.

The potentials of this solid, regarded as homogeneous and of unit density, at an internal and an external point are respectively

$$V_s = 4\pi \left\{\frac{a^2 - b^2}{2} + \Sigma \frac{Y_n'}{2n+1} \frac{r'^n}{a^{n-2}}\right\},$$

$$V_s' = \frac{4\pi}{r'} \left\{\frac{a^3 - b^3}{3} + \Sigma \frac{Y_n'}{2n+1} \frac{a^{n+3}}{r'^n}\right\}.$$

If we differentiate each of these with regard to a, we obtain the potentials of a stratum of unit density bounded by the surfaces whose parameters are a and $a + da$. The actual density of the stratum is $\rho = F(a)$; if then we multiply the differential coefficients by ρ and integrate between the limits $a = a_0$ and $a = a_1$, the required potentials at an internal and external point are found to be

$$V = 4\pi \int \rho \left\{a + \frac{d}{da} \Sigma \frac{Y_n'}{2n+1} \frac{r'^n}{a^{n-2}}\right\} da,$$

$$V' = \frac{4\pi}{r'} \int \rho \left\{a^2 + \frac{d}{da} \Sigma \frac{Y_n'}{2n+1} \frac{a^{n+3}}{r'^n}\right\} da,$$

the limits of the integrals being a_0 and a_1.

We may also find the potential at any point of the solid

defined by the value a of the parameter. In this case the point is external to the strata between a_0 and a and internal to those between a and a_1. The required potential is therefore the sum of the two expressions for V and V', the first between the limits a_0 and a and the second between a and a_1.

The expression thus obtained is the one used by Laplace to find the potential of the earth, regarded as a heterogeneous body, at any internal point.

177. Ex. 1. If the strata of the earth, regarded as solid throughout, are defined by $r = a(1 + Y_2)$, and the density is ga^m, where m is greater than -2, prove that the potential at any internal point is

$$4\pi g\left\{\frac{a^{2+m}}{2+m} + \frac{\mathrm{a}^{3+m} - a^{3+m}}{3+m}\frac{1}{r'} + \frac{Y_2'}{r'^3}\frac{\mathrm{a}^{5+m} - a^{5+m}}{5+m}\right\},$$

where a is the value of a at the boundary.

178. **Solid of revolution.** *To find the potential of a solid of revolution at any point P not occupied by matter.*

Let the axis of the solid be taken as the axis of z with any suitable origin. We have then by Art. 163,

$$V = Y_0 + \frac{Z_0}{r} + Y_1 r + \frac{Z_1}{r^2} + \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Since the attracting body is symmetrical about the axis of z it is evident that V cannot be a function of the angular coordinate ϕ. Hence by Art. 160, $Y_0 = c_0 P_0$, $Z_0 = c_0' P_0$, $Y_1 = c_1 P_1$, &c., where c_0, c_0' &c. are as yet undetermined constants. To find these we put the attracted point on the axis; we then have $P_0 = 1$, $P_1 = 1$, &c. The equation (1) thus becomes

$$V = c_0 + \frac{c_0'}{r} + c_1 r + \frac{c_1'}{r^2} + \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Suppose then we know the potential of the solid at all points of its axis in a convergent series, then (2) is a known series, and therefore the coefficients c_0, c_0' &c. are also known. The series (1) for the potential at P then becomes

$$V = \left(c_0 + \frac{c_0'}{r}\right)P_0 + \left(c_1 r + \frac{c_1'}{r^2}\right)P_1 + \ldots\ldots\ldots\ldots \quad (3).$$

Thus the potential has been found.

In this way we arrive at a theorem of Legendre, viz. *if the attraction of a solid of revolution is known for every external point which is on the prolongation of its axis, it is known for every external point.* See Todhunter's History, Art. 782.

179. It may happen that the expansion (2) giving the

potential at points on the axis takes different forms at different points. Thus when r is less than some quantity a there may be only positive powers of r, and when r is greater than a, there may be only negative powers. Again, if the solid of revolution have a cavity extending to the axis, (2) may assume one form within the cavity and another outside the solid. There will be corresponding changes in the expression (3) giving the potential at P, one form or another holding according as a point Q lies on one part of the axis or another, where Q is such that a point can travel from P to Q without rendering the series (3) divergent or crossing any attracting matter.

If the solid have a ring-like hollow symmetrically placed about the axis of revolution but not extending to it, it is clear that a point P situated in this hollow has no corresponding point Q on the axis from which the potential may be derived. In such a case the values of the constants can be determined when we know the values of the potential along some line passing through the cavity instead of along the axis.

180. Laplace has shown that a Legendre's function may be expressed as a definite integral. We have

$$P_n(p) = \frac{1}{\pi}\int_0^\pi (p \pm \sqrt{p^2-1}\cos\psi)^n \, d\psi.$$

Since p is less than unity, this integral appears to be imaginary. If however we expand the nth power, the integrals of the odd powers of $\cos\psi$ will vanish between the limits, and a real expression for P_n will remain. We may therefore take either of the signs before the radical. There is another integral which may be deduced from (1), viz.

$$P_n(p) = \frac{1}{\pi}\int_0^\pi \frac{d\psi}{(p \pm \sqrt{p^2-1}\cos\psi)^{n+1}}.$$

Suppose that for any portion of the axis the potential is given by $V = f(r)$ where $f(r)$ is such an expansion as (2) Art. 178 with either positive or negative powers of r or both. Substituting for P_n in (3) the first integral in the terms with positive powers of r, and the second integral in those with negative powers, we have

$$V = \frac{1}{\pi}\int_0^\pi f(rp \pm r\sqrt{p^2-1}\cos\psi)\, d\psi.$$

Thus *when the potential is known along the axis in the form*

$V=f(r)$, the potential at other points is known in the form of a definite integral.

181. Ex. 1. To find the potential of a uniform ring of small section at any point not on the axis.

Let the origin be the centre of the ring and let the axis of the ring be the axis of z. Let a be the radius of the ring, M its mass.

The potential at any point Q on the axis distant r from the origin is evidently $M/\sqrt{a^2+r^2}$. We shall expand this in powers of r/a or a/r according as r is less or greater than a. Taking the first supposition, we have

$$V=\frac{M}{a}\left\{1-\frac{1}{2}\left(\frac{r}{a}\right)^2+\frac{1\,.\,3}{2\,.\,4}\left(\frac{r}{a}\right)^4-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\left(\frac{r}{a}\right)^6+\ldots\right\}.$$

When r is greater than a the expression may be deduced from that just written down by interchanging a and r.

The potential of the ring at any point P not on the axis is therefore

$$V=\frac{M}{a}\left\{1-\frac{1}{2}P_2\frac{r}{a}+\frac{1\,.\,3}{2\,.\,4}P_4\left(\frac{r}{a}\right)^4-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}P_6\left(\frac{r}{a}\right)^6+\&c.\right\},$$

$$V'=\frac{M}{r}\left\{1-\frac{1}{2}P_2\frac{a}{r}+\frac{1\,.\,3}{2\,.\,4}P_4\left(\frac{a}{r}\right)^4-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}P_6\left(\frac{a}{r}\right)^6+\&c.\right\},$$

according as r is less or greater than a.

The former of these formulae is useful for instance in finding the attractions of Saturn's ring at a point on the surface of the planet.

Ex. 2. Find the potential of a solid hemisphere at any point of space by means of a definite integral.

The potential of a hemisphere at any point of its axis is given in Art. 66, Ex. 11. The potential at any other point follows at once.

Attraction of Ellipsoids.

182. For the sake of brevity we shall adopt in this section two new terms taken from Thomson and Tait's *Natural Philosophy*.

A *homoeoid* is a shell bounded by two surfaces similar and similarly situated with regard to each other. In what follows we shall somewhat restrict this definition and use the term only when the shell is bounded by concentric ellipsoids.

A *focaloid* is a shell bounded by two confocal ellipsoids.

Thomson and Tait restrict these terms to infinitely thin shells, but it will be more convenient for us to use them in a more general sense, distinguishing the shells as thick or thin according as the thickness is finite or infinitely small.

A shell bounded by two similar and similarly situated surfaces has been called *a homothetic shell* by Chasles in the *Jour. Pol.*, Tome xv., 1837. This is a convenient term when the surfaces are either not concentric or not ellipsoids.

183. Let (a, b, c) be the semi-axes of the internal surface of a thin ellipsoidal shell, $(a + da$, &c.) those of the external surface. Let OPQ be any radius vector drawn from the common centre O cutting the ellipsoids in P and Q, let $OP = r$. Let p be the perpendicular from O on the tangent plane at P, $p + dp$ the perpendicular on a *parallel* tangent plane to the outer ellipsoid. Then dp is equal to the thickness at P.

When the thin shell is a homoeoid we have by the properties of similar figures

$$\frac{da}{a} = \frac{db}{b} = \frac{dc}{c} = \frac{dp}{p} = dk.$$

Since the volume of a solid ellipsoid is $\frac{4}{3}\pi abc$, we find by differentiation that the volume v of the shell is $v = 4\pi abcdk$. Two thin homoeoids are said to be confocal when their inner boundaries are confocal conicoids.

When the shell is a focaloid, we have $a'^2 = a^2 + \lambda$, $b'^2 = b^2 + \lambda$, &c., where (a', b', c') are the semi-axes of the external surface. These give for a thin shell

$$ada = bdb = cdc = pdp = \frac{d\lambda}{2}.$$

The volume v of the shell may be shown by differentiation to be

$$v = \frac{4\pi}{3} \frac{b^2c^2 + c^2a^2 + a^2b^2}{abc} \frac{d\lambda}{2}.$$

184. **Thick homoeoid, internal point.** *To find the potential of a thick homogeneous homoeoid at an internal point.*

It has been shown in Art. 56 that the attraction of such a shell at all internal points is zero. The potential is therefore constant throughout the interior, and it will be sufficient to find the potential at the centre.

Taking polar coordinates with the centre as origin, the mass of any element is $\rho r^2 dr d\omega$, where ρ is the density of the element. The potential V of the whole solid at the centre is therefore $V = \rho \iint r dr d\omega$. If r_1, r_2 be the radii vectores of the two surfaces of the shell, we have $V = \frac{1}{2}\rho \int r_2^2 d\omega - \frac{1}{2}\rho \int r_1^2 d\omega$.

The determination of the potential at the centre of a thick shell bounded by any concentric ellipsoids depends therefore on the evaluation of the integral $\int r^2 d\omega$ taken over the superficies of an ellipsoidal surface.

When the shell is a homoeoid these surfaces are similar. Let

(a, b, c), (ma, mb, mc) be the semi-axes of the external and internal surfaces. We then find $V = \frac{1}{2}\rho(1-m^2)\int r^2 d\omega$, where r is the radius vector of the external boundary.

When the shell is a thin homoeoid, the radial thickness $r_2 - r_1 = kr_2$, where $k = 1 - m$. We then have $V = k\rho\int r^2 d\omega$.

185. To find the integral $\int r^2 d\omega$ we write $d\omega = \sin\theta d\theta d\phi$. Substituting for r^2 its value found from the equation to the ellipsoid, we have

$$\int r^2 d\omega = \iint \frac{\sin\theta d\theta d\phi}{\dfrac{\cos^2\theta}{c^2} + \sin^2\theta\left(\dfrac{\cos^2\phi}{a^2} + \dfrac{\sin^2\phi}{b^2}\right)},$$

where the integration extends over the whole surface of the ellipsoid. Taking only an octant, the limits are $\theta = 0$ to $\theta = \frac{1}{2}\pi$, $\phi = 0$ to $\phi = \frac{1}{2}\pi$. The order of integration is immaterial.

Let us integrate first with regard to ϕ. Dividing both numerator and denominator by $\cos^2\phi$, we find

$$\tfrac{1}{8}\int r^2 d\omega = \iint \frac{\sin\theta d\theta d\tan\phi}{\dfrac{\cos^2\theta}{c^2} + \dfrac{\sin^2\theta}{a^2} + \left(\dfrac{\cos^2\theta}{c^2} + \dfrac{\sin^2\theta}{b^2}\right)\tan^2\phi}.$$

By obvious processes in the integral calculus

$$\int_0^\infty \frac{dt}{A + Bt^2} = \left[\frac{1}{\sqrt{(AB)}}\tan^{-1} t\sqrt{\frac{B}{A}}\right]_0^\infty = \frac{\pi}{2\sqrt{(AB)}}.$$

It therefore follows that

$$\tfrac{1}{8}\int r^2 d\omega = \frac{\pi}{2}\cdot\int \frac{\sin\theta d\theta}{\sqrt{\left(\dfrac{\cos^2\theta}{c^2} + \dfrac{\sin^2\theta}{a^2}\right)}\sqrt{\left(\dfrac{\cos^2\theta}{c^2} + \dfrac{\sin^2\theta}{b^2}\right)}}.$$

To interpret this expression, let us produce the radius vector OP or r to cut the tangent plane drawn at the extremity C of the axis of z. Let R be the point of intersection and let $CR = u$, then $u = c\tan\theta$. Since the limits of θ are 0 and $\frac{1}{2}\pi$, those of u are 0 and ∞. Substituting, we find

$$\int r^2 d\omega = 2\pi abc\int_0^\infty \frac{du^2}{(a^2+u^2)^{\frac{1}{2}}(b^2+u^2)^{\frac{1}{2}}(c^2+u^2)^{\frac{1}{2}}},$$

where the integration on the left side extends over the whole surface of the ellipsoid.

186. If we write $I = \displaystyle\int_0^\infty \frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}$, we find that the potential of a thick homoeoid is

$$V = \rho(1-m^2)\pi abc\,.\,I = \tfrac{3}{4}MI\frac{1-m^2}{1-m^3}.$$

The potential of a thin homoeoid is $V=\frac{1}{2}M\,.\,I$, where, in each case, M is the mass of the attracting body.

187. Ex. 1. If we reverse the order of integration in Art. 185 and integrate first with regard to ϕ, we find $\qquad \int r^2 d\omega = 8abc\int \tan^{-1}\frac{W}{c}\frac{d\phi'}{W}$,

where $W^2=(a^2-c^2)\cos^2\phi'+(b^2-c^2)\sin^2\phi'$ and $\tan\phi=\frac{b}{a}\tan\phi'$. The integration on the left-hand side extends over the whole ellipsoid and that on the right-hand side from $\phi'=0$ to $\frac{1}{2}\pi$.

Ex. 2. Show that the integral I, and therefore the potential V, may be expressed as an elliptic integral. Thus $\qquad I=\frac{2}{\sqrt{(c^2-b^2)}}\int\frac{d\psi}{\sqrt{(1-\lambda\sin^2\psi)}}$,

where $\lambda=\frac{c^2-a^2}{c^2-b^2}$ and the limits are $\psi=0$ to $\psi=\sqrt{\frac{c^2-b^2}{c^2}}$. The integral is real if the axis chosen, as that of c, is the longest of the three.

Ex. 3. The ellipsoid becomes a spheroid when $a=b$. Show that the integral I becomes $\qquad \frac{2}{(c^2-a^2)^{\frac{1}{2}}}\log\frac{c+(c^2-a^2)^{\frac{1}{2}}}{a}$ or $\frac{2}{(a^2-c^2)^{\frac{1}{2}}}\sin^{-1}\left(\frac{a^2-c^2}{a^2}\right)^{\frac{1}{2}}$,
according as the spheroid is prolate or oblate.

Ex. 4. Show that $\qquad \left(\frac{d}{da^2}+\frac{d}{db^2}+\frac{d}{dc^2}\right)I=-\frac{1}{abc}$,

$$\left(a^2\frac{d}{da^2}+b^2\frac{d}{db^2}+c^2\frac{d}{dc^2}+\frac{1}{2}\right)I=0, \qquad 2(a^2-b^2)\frac{d^2I}{da^2db^2}=\frac{dI}{da^2}-\frac{dI}{db^2}.$$

Ex. 5. Assuming that for an ellipsoid $\int r^2 d\omega=2\pi abcI$,

prove $\qquad \int z^2 d\omega=-4\pi abc\,.\,c^2\frac{dI}{dc^2}, \qquad \int r^2z^2d\omega=2\pi abc\left(c^4\frac{dI}{dc^2}+\frac{c^2}{2}I\right)$,

$$\int r^4 d\omega=2\pi abc\left(a^4\frac{dI}{da^2}+b^4\frac{dI}{db^2}+c^4\frac{dI}{dc^2}+\frac{a^2+b^2+c^2}{2}I\right).$$

Ex. 6. If $\int r^{2m}d\omega=abcR_m$, prove that

$$\left\{a^2\frac{d}{da^2}+b^2\frac{d}{db^2}+c^2\frac{d}{dc^2}+\left(\tfrac{3}{2}-m\right)\right\}R_m=0,$$

$$\left\{a^4\frac{d}{da^2}+b^4\frac{d}{db^2}+c^4\frac{d}{dc^2}+\frac{a^2+b^2+c^2}{2}\right\}R_m=mR_{m+1},$$

$$\int r^{2m}z^2 d\omega=\frac{abc^3}{m}\left\{\tfrac{1}{2}R_m+c^2\frac{dR_m}{dc^2}\right\}.$$

Ex. 7. Instead of the standard integral represented by I we may use the integral $\qquad J=\int_0^\infty\frac{abc\,du}{u(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}$.

We then have $\qquad \frac{dJ}{da}=-\frac{bc}{a}\frac{dI}{da}, \qquad \frac{dJ}{db}=-\frac{ca}{b}\frac{dI}{db}$, &c.

If we write α, β, γ for the *reciprocals* of a^2, b^2, c^2 we easily find

$$\int z^2 d\omega=-4\pi\frac{dJ}{d\gamma}, \qquad \int r^2 d\omega=-4\pi\left(\frac{d}{d\alpha}+\frac{d}{d\beta}+\frac{d}{d\gamma}\right)J,$$

$$\int x^{2i}y^{2j}z^{2k}d\omega=\frac{4\pi(-1)^{i+j+k}}{\Gamma(i+j+k)}\left(\frac{d}{d\alpha}\right)^i\left(\frac{d}{d\beta}\right)^j\left(\frac{d}{d\gamma}\right)^k J,$$

where the integrations extend over the surface of an ellipsoid whose principal diameters are the axes of coordinates. The symbol $\Gamma(n)$ represents, as usual, the product of the natural numbers up to but excluding n.

188. **Theorems on thin homoeoids.** *The potential at any internal point of a thin homoeoid being known it is required to find the potential at any external point.*

Let two confocal ellipsoids have for their semi-axes (a, b, c), (a', b', c'); points on these are said to correspond when their coordinates are connected by the relations

$$\frac{x}{a}=\frac{x'}{a'}, \qquad \frac{y}{b}=\frac{y'}{b'}, \qquad \frac{z}{c}=\frac{z'}{c'} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let $d\sigma$, $d\sigma'$ be two triangular elements of area at P, P' such that the corners are corresponding points; let p, p' be the perpendiculars from the centre O on the tangent planes. The volumes of the tetrahedra whose bases are $d\sigma$, $d\sigma'$ and common vertex O are respectively $\frac{1}{3}pd\sigma$ and $\frac{1}{3}p'd\sigma'$. The first of these volumes is expressed by one sixth of the determinant in the margin $\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$ where the several rows express the coordinates of the corners. The second volume is expressed in the same way with accented letters to represent the corresponding points on the second ellipsoid. It is evident from the relations (1) that these determinants are in the ratio $abc : a'b'c'$. We therefore infer that the elements of surface of the two confocals are connected by the equation

$$\frac{pd\sigma}{p'd\sigma'}=\frac{abc}{a'b'c'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Since any elementary areas at P and P' may be subdivided into triangles, it is evident that this relation holds for elementary areas $d\sigma$, $d\sigma'$ of any shape, provided only their boundaries are formed by corresponding points.

Since the thickness of a thin homoeoid is represented by kp, it follows that *the volumes of corresponding elements of two confocal thin homoeoids are in a constant ratio.* Adding these elementary volumes together, it is easily seen that *this constant ratio is equal to that of the whole volumes of the two shells.* If the shells are of such thicknesses that their whole volumes are equal, then the volumes of all corresponding elements are also equal.

189. We shall now require the following geometrical theorem:—*the distances between two points one on each of two confocal*

ellipsoids is equal to the distance between their corresponding poin A proof may be found in Smith's *Solid Geometry*, Art. 166. Tl theorem is usually called Ivory's theorem after its discoverer, w also applied it to determine the potential of an ellipsoid at external point.

Let P, P' be two corresponding points, one on each of t confocal thin homoeoids of equal volume; let also Q, Q' be any t corresponding elementary volumes each equal to dv. Let t equal distances PQ', $P'Q$ be represented by R. If $f'(R)$ represe the law of attraction, the potentials at P and P' of these el mentary *volumes* are each $f(R)\,dv$. Integrating over the whc surfaces of the shells, we see that *the potential of the inner th homoeoid at the external point P' is equal to that of the outer th homoeoid at the corresponding internal point P, provided t densities are equal at corresponding points**.

Thus when the potentials of thin homoeoids at all intern points are known, their potentials at all external points are al known.

190. It is evident that *the potentials of these shells are equ whatever be the law of attraction* provided the potential is a functic of the distance only.

191. *The potentials are also equal if the shells are heterogeneou and the density at any point is a function of $(x/a, y/b, z/c)$.* In th case it is evident that the densities of the shells are equal a corresponding points.

192. The theorem may also be used (though not so simpl to compare the potentials even when the density is any functio of the coordinates. It will be convenient to express this result i an analytical form.

Let the density ρ of a thin homoeoid (semi-axes a, b, c) b $f(x, y, z)$, and let v be the volume of the shell. It is require to find its potential at any external point (ξ', η', ζ'). Let

* Chasles in his *Nouvelle solution du problème de l'attraction d'un ellipsoïc hétérogène sur un point extérieur*, Liouville, vol. v. 1840, shows that thin confoc homoeoids have potentials at corresponding points proportional to their masses, bu considers only the case in which they are homogeneous. Knowing that th potential of the outer at an internal point is constant, he deduces several theorem on the attractions of the inner shell at external points. He finds the attraction o a solid heterogeneous ellipsoid by dividing it into thin elementary homoeoids, th strata of equal density being the elementary homoeoids. The case in which th homoeoid is heterogeneous is not discussed.

confocal ellipsoid be described passing through the point (ξ', η', ζ') so that its major axis is given by the equation

$$\frac{\xi'^2}{a'^2} + \frac{\eta'^2}{a'^2 - (a^2 - b^2)} + \frac{\zeta'^2}{a'^2 - (a^2 - c^2)} = 1 \qquad (1).$$

Let this ellipsoid be the inner boundary of a second thin homoeoid whose volume is equal to that of the former. Let its density at any point (x', y', z') be $\rho' = f\left(\frac{a}{a'}x', \frac{b}{b'}y', \frac{c}{c'}z'\right)$. The potential of this second homoeoid at the internal point $\left(\frac{a}{a'}\xi', \frac{b}{b'}\eta', \frac{c}{c'}\zeta'\right)$ is equal to the potential required.

193. Taking the case in which the two thin homoeoids are homogeneous, the potential of the outer has been proved constant for all internal points, Art. 56. It immediately follows that the potential of the inner is the same at all external points which lie on the same confocal. We therefore infer that *the level surfaces of any thin homogeneous homoeoid are confocal ellipsoids.*

194. Since two thin confocal homoeoids have the same level surfaces, their potentials can be made equal over any level surface enclosing both by properly adjusting their masses. It immediately follows that their potentials are also equal throughout all external space, Art. 106. Since the potentials of finite bodies vanish at infinity in the ratio of their masses, it is evident that the masses of the two homoeoids must be equal. We have therefore the following theorem, *the potentials, and therefore also the resolved attractions, of two confocal thin homoeoids of equal masses are equal throughout all space external to both.*

195. **Lines of force.** The lines of force of a homogeneous thin homoeoid are the orthogonal trajectories of all the confocal ellipsoids. Let (a', b', c'), (a'', b'', c''), (a''', b''', c''') be the semi-axes of the confocal ellipsoid and hyperboloids which pass through any external point (ξ', η', ζ'). Then by a theorem in solid geometry

$$\xi' = \frac{a'a''a'''}{\sqrt{(a^2 - b^2)(a^2 - c^2)}}, \quad \eta' = \frac{b'b''b'''}{\sqrt{(b^2 - a^2)(b^2 - c^2)}}, \quad \zeta' = \frac{c'c''c'''}{\sqrt{(c^2 - a^2)(c^2 - b^2)}},$$

see Salmon's *Solid Geometry*, Art. 160. Since these conicoids intersect at right angles the curve of intersection of the two hyperboloids is an orthogonal trajectory of all the confocal ellipsoids. The

required trajectories are therefore found by regarding (a'', b'', c'') and (a''', b''', c''') as constants. It follows that $\frac{\xi'}{a'}, \frac{\eta'}{b'}, \frac{\zeta'}{c'}$ are constant for the same orthogonal. Thus it appears that *any line of force of a homogeneous thin homoeoid intersects all the confocal ellipsoids in corresponding points.*

196. **Thin homoeoid, external point.** *To find the potential of a homogeneous thin homoeoid at an external point.*

Let (ξ', η', ζ') be the external point. The potential V of the given homoeoid at this point has been proved equal to that of a second homoeoid (of equal mass and passing through (ξ', η', ζ') at the internal point $\left(\frac{a}{a'}\xi', \frac{b}{b'}\eta', \frac{c}{c'}\zeta'\right)$, see Art. 192. It immediately follows from Art. 186 that

$$V = \frac{M}{2}\int_0^\infty \frac{du}{(a'^2+u)^{\frac{1}{2}}(b'^2+u)^{\frac{1}{2}}(c'^2+u)^{\frac{1}{2}}},$$

where M is the mass of the homoeoid.

This integral may be put into another form, which contains the semi-axes (a, b, c) instead of (a', b', c'). Putting $a'^2 = a^2 + \lambda$, $b'^2 = b^2 + \lambda$, $c'^2 = c^2 + \lambda$ and $u' = u + \lambda$, we have

$$V = \frac{M}{2}\int_\lambda^\infty \frac{du'}{(a^2+u')^{\frac{1}{2}}(b^2+u')^{\frac{1}{2}}(c^2+u')^{\frac{1}{2}}},$$

where the lower limit λ is determined by the equation

$$\frac{\xi'^2}{a^2+\lambda} + \frac{\eta'^2}{b^2+\lambda} + \frac{\zeta'^2}{c^2+\lambda} = 1.$$

197. *To find the resultant attraction of a thin homogeneous homoeoid at an external point.*

Since the level surfaces of the homoeoid are confocal quadrics, the resultant attraction at the point ξ', η', ζ', is normal to the confocal which passes through that point. If p' be the perpendicular from the centre on the tangent plane to the confocal, the resultant force F may be found by differentiating the expression for V found in the last article with regard to p'. We therefore have, by Art. 183,

$$F = \frac{dV}{d\lambda}\frac{d\lambda}{dp'} = -\frac{M}{2}\frac{1}{(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}}\cdot 2p' = -\frac{Mp'}{a'b'c'}.$$

198. When the attracted point is close to the external surface of the shell, the semi-axes a', b', c' of the confocal become a, b, c.

By Art. 183 the expression for F then becomes

$$F = -4\pi\rho dp = -4\pi\sigma,$$

where ρ is the volume density and σ the surface density of the thin shell. Thus *the resultant attraction of a thin homoeoid at any point just outside the surface is equal to twice that of an infinite plate of the same thickness as the shell at that point.* See Art. 59.

199. Conversely, we may use Green's theorem on the attraction of a thin stratum to find the attraction of a homoeoid at any external point.

It has been shown in Art. 194 that if we describe a second homoeoid external to the given one of equal mass its potential will be the same as that of the given homoeoid throughout all external space. Let the point (ξ', η', ζ') at which the attraction is required be just outside this second homoeoid. Then by Green's theorem, its normal attraction at the point is $4\pi\sigma$, where σ is the surface density, Art. 117. Since a confocal is a level surface, the normal attraction is the same as the resultant attraction.

Let M be the mass of either homoeoid, ρ the density, then $\sigma = \rho p' dk$ and $M = 4\pi a'b'c'\rho dk$; see Art. 183. The required resultant attraction F is therefore $$F = M\frac{p'}{a'b'c'}.$$
Here a', b', c' are the semi-axes of a confocal through the attracted point, and p' is the length of the perpendicular drawn from the centre on the tangent plane at that point.

This expression for the resultant force is given by Chasles in the *Journal Polytechnique*, 1837, Tome xv. See also the *Quarterly Journal*, 1867.

200. Ex. 1. Deduce from the expression for the resultant force F the value of the potential V at any external point.

We have $-\dfrac{dV}{dp'} = F = M\dfrac{p'}{a'b'c'}$. If $a'^2 - a^2 = \lambda$, we have by well-known properties of confocals $p'dp' = \frac{1}{2}d\lambda$. Substituting, we find $V = -\dfrac{M}{2}\displaystyle\int_\infty^\lambda \frac{d\lambda}{a'b'c'}$ together with an undetermined constant. Since V vanishes at an infinite distance from the shell, i.e. at all points given by $\lambda = \infty$, we see that the constant is zero.

Ex. 2. The attractions of a given thin homoeoid on two corresponding elementary areas taken on any two confocal ellipsoids are equal.

[Chasles, *Journal Polytechnique*, 1837, Tome xv.]

Ex. 3. The attraction of a thin homoeoid at any point situated on its external surface is proportional to the thickness of the shell at that point. [Chasles.]

Ex. 4. A thin prolate spheroidal shell of mass M is divided into two portions by a diametral plane perpendicular to its axis. Prove that the pressure per unit of length on the line of separation, due to the mutual attraction of the parts, is

$$\frac{M^2}{8\pi b}\frac{\log a - \log b}{a^2 - b^2}.$$ [Math. Tripos.]

201. **Potential of a solid ellipsoid.** *To find the potential of a solid homogeneous ellipsoid at any external point P' whose coordinates are (ξ', η', ζ').*

Let us take as an element a thin homoeoid having its surfaces *similar* to that of the surface of the given solid. Let a, b, c be the semi-axes of the surface of the solid, ma, mb, mc those of the inner surface of the elementary homoeoid. Let also $m^2a^2 + \lambda$, $m^2b^2 + \lambda$, $m^2c^2 + \lambda$ be the squares of the semi-axes of the elliptic *confocal* drawn through P', then λ is given by the cubic

$$\frac{\xi'^2}{m^2a^2 + \lambda} + \frac{\eta'^2}{m^2b^2 + \lambda} + \frac{\zeta'^2}{m^2c^2 + \lambda} = 1 \quad\text{.........}\quad (1).$$

The volume of the shell bounded by the ellipsoids m and $m + dm$ may be obtained by differentiating $\frac{4}{3}\pi abcm^3$ and is therefore $4\pi abcm^2dm$. If the density is ρ, its potential at P' is

$$2\pi\rho abcm^2dm\int_\lambda^\infty \frac{du}{R} \quad\text{..................}\quad (2),$$

where $R^2 = (m^2a^2 + u)(m^2b^2 + u)(m^2c^2 + u)$. The potential of the whole solid is found by integrating (2) between the limits $m = 0$ and $m = 1$.

To simplify both the equation (1) and the expression for R we put $\lambda = m^2\mu$ and $u = m^2v$. The potential V of the whole solid is now given by

$$\frac{V}{2\pi\rho abc} = \int_0^1 mdm\int_\mu^\infty \frac{dv}{U} \quad\text{..................}\quad (3),$$

where μ is determined by the equation

$$\frac{\xi'^2}{a^2 + \mu} + \frac{\eta'^2}{b^2 + \mu} + \frac{\zeta'^2}{c^2 + \mu} = m^2 \quad\text{.............}\quad (4),$$

and
$$U^2 = (a^2 + v)(b^2 + v)(c^2 + v).$$

Integrating (3) by parts we have

$$\frac{V}{\pi\rho abc} = m^2\int_\mu^\infty \frac{dv}{U} - \int m^2 d\left[\int_\mu^\infty \frac{dv}{U}\right] \quad\text{.............}\quad (5).$$

The first of the two terms on the right-hand side is to be taken between the limits $m = 0$ and $m = 1$. Referring to (4) we see that $\mu = \infty$ when $m = 0$; let $\mu = \epsilon$ be the value of μ when $m = 1$, then $a^2 + \epsilon$, $b^2 + \epsilon$, $c^2 + \epsilon$ are the squares of the semi-axes

of an ellipsoid drawn through P' confocal to the surface of the attracting body. The first term therefore reduces to $\int_\epsilon^\infty \frac{dv}{U}$.

The second term of (5) is the same as $-\int m^2 \frac{-d\mu}{U'}$, where U' differs from U only in having μ written for v. This term also is to be taken between the limits $m=0$ and $m=1$. The two terms together therefore reduce to

$$\frac{V}{\pi\rho abc} = \int_\epsilon^\infty \frac{dv}{U} + \int_\infty^\epsilon \frac{m^2 d\mu}{U'} \quad\text{......(6).}$$

Writing u for v and μ in these expressions to obtain uniformity in the notation, and substituting for m^2 from (4), we have*

$$\frac{V}{\pi\rho abc} = \int_\epsilon^\infty \left(1 - \frac{\xi'^2}{a^2+u} - \frac{\eta'^2}{b^2+u} - \frac{\zeta'^2}{c^2+u}\right) \frac{du}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}.$$

202. *This expression for the potential may be put into another form in which both the limits of the integral are constants.*

Let $u=\epsilon+v$ and let also $a'^2=a^2+\epsilon$, $b'^2=b^2+\epsilon$, $c'^2=c^2+\epsilon$; the expression then takes the form

$$\frac{V}{\pi\rho abc} = \int_0^\infty \left(1 - \frac{\xi'^2}{a'^2+v} - \frac{\eta'^2}{b'^2+v} - \frac{\zeta'^2}{c'^2+v}\right) \frac{dv}{(a'^2+v)^{\frac{1}{2}}(b'^2+v)^{\frac{1}{2}}(c'^2+v)^{\frac{1}{2}}},$$

where (a', b', c') are now the semi-axes of the confocal through the attracted point.

Let $I' = \int_0^\infty \frac{du}{(a'^2+u)^{\frac{1}{2}}(b'^2+u)^{\frac{1}{2}}(c'^2+u)^{\frac{1}{2}}}$, so that I' differs from the value of I given in Art. 186 in having a', b', c' written for a, b, c.

We now have $\quad V = \frac{3}{4}M.\left(I' + 2\frac{dI'}{da'^2}\xi'^2 + 2\frac{dI'}{db'^2}\eta'^2 + 2\frac{dI'}{dc'^2}\zeta'^2\right),$

where $M = \frac{4}{3}\pi\rho abc$ and is therefore the mass of the attracting ellipsoid.

This expression for V is sometimes written in the form

$$V = -\tfrac{1}{2}(A'\xi'^2 + B'\eta'^2 + C'\zeta'^2) + D',$$

where A', B', C', D' are functions of (ξ', η', ζ') which are constant over the surface of any confocal ellipsoid.

* Lejeune Dirichlet expresses in this form the attraction of an ellipsoid at both an internal and an external point, without claiming it as new. See Crelle's *Journal*, vol. XXXII. 1846. But Poisson's forms for the axial components are identical with the differential coefficients of V. *Mémoires de l'Institut*, vol. XIII. 1833. Both Todhunter (see *History*) and Cayley (*Quarterly Journal*, vol. II.) appear to connect these forms with the name of Rodrigues, 1815. They have also been ascribed to Jacobi.

203. We notice that the axes (a, b, c) of the attracting ellipsoid have disappeared from the right-hand side of the expression for V/M except so far as they are contained in (a', b', c'). This ratio is therefore the same for all attracting ellipsoids whose external boundaries are confocals. We thus deduce Maclaurin's theorem, viz. *the potentials of two solid homogeneous ellipsoids bounded by confocals at any point external to both are proportional to their masses.*

Since we may regard a focaloid as the difference of two ellipsoids, it is obvious that the same theorem will apply to focaloids also. *The potentials of confocal thick focaloids at any external point are proportional to their masses.*

It follows from the properties of a potential, that *all homogeneous confocal focaloids attract the same external point with forces whose directions are the same and whose magnitudes are proportional to the masses of the attracting bodies.*

204. Ex. 1. The attraction of a thin homoeoid at any external point is the same as that of a thin disc bounded by its elliptic focal conic and having the surface density at any point P *inversely* proportional to $\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}$, where (x, y) are the coordinates of the point P referred to the axes of the focal conic.

This follows from the theorem in Art. 194, since the disc may be regarded as a confocal homoeoid in which the axis c is evanescent. To find its law of density we notice that the mass on any elementary area $dxdy$ is $2\rho \frac{dz}{dc} dxdydc$. Now

$$z^2 = c^2 - \frac{c^2}{a^2}x^2 - \frac{c^2}{b^2}y^2,$$

and the surfaces being similar, c/a and c/b are constants. Hence $\frac{dz}{dc} = \frac{c}{z}$. The result then follows immediately.

Ex. 2. The attraction of a solid ellipsoid at any external point is the same as that of a thin disc bounded by its elliptic focal conic and having its density at any point *directly* proportional to $\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}$.

Use Maclaurin's theorem, Art. 203.

Ex. 3. The attraction of a thin prolate spheroidal homoeoid at any external point is the same as that of a thin homogeneous straight rod joining the foci.

This result may be deduced from that given in Art. 203, but it follows more easily from Art. 106. The thin shell and the straight line have the same level surfaces (viz. confocal conicoids), hence their attractions are also the same.

Ex. 4. The attraction of a solid prolate spheroid at any external point is the same as that of a straight rod joining the foci, and having its line density at any point P proportional to $SP \,.\, PH$.

Ex. 5. Investigate Legendre's expression for the x component of the attraction of a solid ellipsoid at the external point (ξ', η', ζ'), viz.

$$X = \frac{3M\xi'}{a'} \int_0^1 \frac{x^2 dx}{(a'^2 + (b^2 - a^2) x^2)^{\frac{1}{2}} (a'^2 + (c^2 - a^2) x^2)^{\frac{1}{2}}},$$

where a' is the semi-major axis of the confocal through the attracted point and M is the mass of the shell.

Put $$x^2 = a'^2/(u + a^2).$$

Ex. 6. If Q be any point of a solid ellipsoid, OQR the semi-diameter drawn through Q, and if the density at Q be $K\{OQ/OR\}^n$, prove that the potential at any external point (ξ', η', ζ') is

$$\frac{2\pi abcK}{n+2} \int_0^\infty \left\{1 - \left(\frac{\xi'^2}{a'^2 + v} + \&c.\right)^{\frac{n+2}{2}}\right\} \frac{dv}{(a'^2+v)^{\frac{1}{2}} (b'^2+v)^{\frac{1}{2}} (c'^2+v)^{\frac{1}{2}}}.$$

205. *To find the potential of a solid homogeneous ellipsoid, axes a, b, c, at any internal point P.*

We follow the same line of argument as that adopted in the case of spheres. Through P we pass an ellipsoid concentric with and similar to the external surface; thus dividing the solid into two parts. Let ma, mb, mc be the semi-axes of this ellipsoid. The potentials at P of the outer and inner portions are respectively

$$\pi\rho abc\,(1 - m^2) \int_0^\infty \frac{du}{(a^2+u)^{\frac{1}{2}} (b^2+u)^{\frac{1}{2}} (c^2+u)^{\frac{1}{2}}},$$

$$\pi\rho abc\, m^3 \int_0^\infty \frac{dv}{(m^2a^2+v)^{\frac{1}{2}} (m^2b^2+v)^{\frac{1}{2}} (m^2c^2+v)^{\frac{1}{2}}} \left\{1 - \frac{\xi^2}{m^2a^2+v} - \&c.\right\},$$

where as before ρ is the density of the ellipsoid and (ξ, η, ζ) the coordinates of the attracted point.

Writing $v = m^2u$ in the second of these and adding the two together, we find

$$\frac{V}{\pi\rho abc} = \int_0^\infty \frac{du}{(a^2+u)^{\frac{1}{2}} (b^2+u)^{\frac{1}{2}} (c^2+u)^{\frac{1}{2}}} \left\{1 - \frac{\xi^2}{a^2+u} - \frac{\eta^2}{b^2+u} - \frac{\zeta^2}{c^2+u}\right\}.$$

We may put this result in another form which is sometimes more convenient. As before, writing

$$I = \int_0^\infty \frac{du}{(a^2+u)^{\frac{1}{2}} (b^2+u)^{\frac{1}{2}} (c^2+u)^{\frac{1}{2}}},$$

we have $$\frac{V}{\pi\rho abc} = I + 2\frac{dI}{da^2}\xi^2 + 2\frac{dI}{db^2}\eta^2 + 2\frac{dI}{dc^2}\zeta^2.$$

This is also sometimes written in the form

$$V = -\tfrac{1}{2}(A\xi^2 + B\eta^2 + C\zeta^2) + D,$$

where A, B, C, D are the same functions of a, b, c that A', B', C', D' as used in Art. 202, are of a', b', c'.

206. Ex. 1. Show that

$$A+B+C=4\pi\rho, \qquad Aa^2+Bb^2+Cc^2-2D=0.$$

The first of these follows from Poisson's theorem, Art. 80. Both follow at once from the results of Ex. 4, Art. 187.

Ex. 2. Show that $\frac{A}{a}da+\frac{B}{b}db+\frac{C}{c}dc$ is a perfect differential.

[Townsend's Theorem, *Quarterly Journal*, Vol. XII. p. 70.]

Ex. 3. If D be the potential at the centre of the ellipsoid, show that the potential at a point P distant r from the centre and situated on the straight line $\xi/a=\eta/b=\zeta/c$ is

$$V=D\left(1-\frac{r^2}{a^2+b^2+c^2}\right).$$

If the point P be situated on a straight line making equal angles with the axes, the potential at P is $V=D-\frac{2}{3}\pi\rho r^2$.

Ex. 4. If V_s be the potential of a thin focaloid at an internal point P, prove

$$V_s=\frac{\delta v}{v}V-\pi\rho\lambda\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right),$$

where v is the volume enclosed by the shell, δv that of the shell itself, V is the potential at the same point of the enclosed volume supposed to be of the same density as the shell itself, and λ is the difference of the squares of the semi-axes of the two boundaries of the shell. See Art. 183.

To prove this we notice that for a solid ellipsoid we have

$$\frac{V}{\pi\rho abc}=I+2\frac{dI}{da^2}\xi^2+\&c.,$$

as in Art. 205. To deduce the potential of a thin focaloid we find δV on the supposition that a^2, b^2, c^2 are each increased by the same quantity λ. This is evidently effected by performing on both sides of the equation, as it stands, the operation

$$\delta=\lambda\left(\frac{d}{da^2}+\frac{d}{db^2}+\frac{d}{dc^2}\right).$$

The result follows at once from Ex. 4, Art. 187.

Ex. 5. Show that the potential of a thin focaloid at an external point is $\frac{\delta v}{v}V$.

207. **Attraction of a solid ellipsoid.** *To find the attraction of a solid ellipsoid at an internal point P.*

The axial components of the attraction may be deduced from the value of the potential found in Art. 205, but the following process is so simple as to merit attention.

Through P we pass an ellipsoid concentric with and similar to the boundary of the solid. The attraction at P of the portion of the solid external to this ellipsoid has been proved to be zero in Art. 56. It is therefore necessary only to find the attraction at P of the portion of the solid bounded by this ellipsoid. The problem is thus reduced to that of finding the attraction of an ellipsoid at a point on its surface. Let the semi-axes of this ellipsoid be ma, mb, mc.

We now construct an elementary cone whose vertex is P and whose base is an element of the surface. If $d\omega$ be the solid angle of the cone, its attraction at P is $\int \frac{\rho r^2 d\omega dr}{r^2}$ taken between the limits $r = 0$ and $r = r$. The attraction is therefore $\rho r d\omega$.

The axial components of the attraction of the whole ellipsoid at P are therefore

$$X = -\rho \int r\lambda d\omega, \qquad Y = -\rho \int r\mu d\omega, \qquad Z = -\rho \int r\nu d\omega \text{......(1)},$$

where (λ, μ, ν) are the direction cosines of the radius vector r drawn from P as origin, and the integrations are to be taken so as to include all the elementary cones which lie on one side of the tangent plane at P.

If (ξ, η, ζ) be the coordinates of P when referred to the centre, the equation of the ellipsoid becomes

$$\frac{(\xi - \lambda r)^2}{m^2 a^2} + \frac{(\eta - \mu r)^2}{m^2 b^2} + \frac{(\zeta - \nu r)^2}{m^2 c^2} = 1 \text{........(2)}.$$

But since the point (ξ, η, ζ) lies on the surface this gives

$$r = 2\,\frac{\dfrac{\xi\lambda}{a^2} + \dfrac{\eta\mu}{b^2} + \dfrac{\zeta\nu}{c^2}}{\dfrac{\lambda^2}{a^2} + \dfrac{\mu^2}{b^2} + \dfrac{\nu^2}{c^2}} \text{ .... (3)}.$$

This value of r has to be substituted in the expressions (1) and the integrations effected. As the radius vector turns round P, it is evident by (3) that no values of λ, μ, ν make r imaginary. Since the value of r determined by λ, μ, ν differs only in sign from that determined by $-\lambda, -\mu, -\nu$, the equation (3) represents the surface twice over. If then we integrate the equations (1) taking *all* positions of the radius vector and not merely those on one side of the tangent plane, we shall obtain in each case twice the required attraction.

We therefore have $$X = -\rho \int \frac{\dfrac{\xi\lambda^2}{a^2} + \dfrac{\eta\lambda\mu}{b^2} + \dfrac{\zeta\lambda\nu}{c^2}}{\dfrac{\lambda^2}{a^2} + \dfrac{\mu^2}{b^2} + \dfrac{\nu^2}{c^2}}\, d\omega,$$

where (λ, μ, ν) have all possible values. It is obvious that the term containing the product $\lambda\mu$ disappears on integration, for the elements corresponding to (λ, μ) and $(\lambda, -\mu)$ destroy each other. In the same way the term containing the product $\lambda\nu$ disappears.

We therefore have

$$X = -\rho\xi\int\frac{\frac{\lambda^2}{a^2}d\omega}{\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}},\qquad Y = -\rho\eta\int\frac{\frac{\mu^2}{b^2}d\omega}{\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}},\qquad Z = \&c.$$

These are usually written in the form

$$X = -A\xi,\quad Y = -B\eta,\quad Z = -C\zeta.$$

We notice that *the constants A, B, C are functions of the ratios of the axes* and are therefore the same for all similar ellipsoids.

The integrals given above for A, B, C may also be written in the form $\quad A = \rho\int\frac{x^2}{a^2}d\omega,\quad B = \rho\int\frac{y^2}{b^2}d\omega,\quad C = \rho\int\frac{z^2}{c^2}d\omega,$

where the integration extends over the whole surface of the ellipsoid.

It easily follows that

$$A + B + C = 4\pi\rho$$
$$Aa^2 + Bb^2 + Cc^2 = \rho\int r^2 d\omega,$$

where r is the radius vector of the bounding ellipsoid drawn from the centre as origin.

It is evident that A, B, C have here the same meaning as in Art. 205. See Art. 187, Ex. 4.

208. Ex. 1. Find the attraction of the spheroid whose semi-axes are a, a, c at an internal point.

If (ξ, η, ζ) be the point, the required attractions are $X = -A\xi$, $Y = -A\eta$, $Z = -C\zeta$, where A and C are given by

$$2A + C = 4\pi\rho \quad\text{.. (1)},$$

$$2Aa^2 + Cc^2 = \rho\iint\frac{\sin\theta\, d\theta\, d\phi}{\frac{\cos^2\theta}{c^2}+\frac{\sin^2\theta}{a^2}} \quad\text{.......................... (2)}.$$

The limits of integration are $\theta = 0$ to $\theta = \pi$ and $\phi = 0$ to $\phi = 2\pi$. Writing $\cos\theta = z$, this reduces to

$$2Aa^2 + Cc^2 = -2\pi\rho a^2c^2\int\frac{dz}{c^2 + (a^2 - c^2)z^2},$$

where the limits are $z = 1$ to $z = -1$.

If the spheroid is oblate, a is greater than c, and

$$2Aa^2 + Cc^2 = \frac{4\pi\rho a^2 c}{\sqrt{(a^2 - c^2)}}\tan^{-1}\frac{\sqrt{(a^2 - c^2)}}{c}.$$

If the spheroid is prolate, a is less than c, and

$$2Aa^2 + Cc^2 = \frac{4\pi\rho a^2 c}{\sqrt{(c^2 - a^2)}}\log\frac{c + \sqrt{(c^2 - a^2)}}{a} \quad\text{.................... (3)}.$$

Ex. 2. Show that an attracting homogeneous oblate spheroid of eccentricity $\frac{1}{2}$, in the centre of which there acts a repulsive force μr, will have its own surface for one of its level surfaces if $3\mu = 8\pi\rho(5\pi\sqrt{3} - 27)$. [Coll. Ex. 1888.]

Ex. 3. If a concentric ellipsoidal cavity be cut out of a solid homogeneous sphere, show that within the cavity the equipotential surfaces are given by

$$(2A-B-C)\,x^2+(2B-C-A)\,y^2+(2C-A-B)\,z^2=\text{constant},$$

and A, B, C are constants depending on the *shape* of the cavity.

[St John's Coll. 1887.]

Ex. 4. A homogeneous ellipsoid attracts a body M according to the law of the inverse square; prove that if M be a spherical or cubical portion of the mass of the ellipsoid itself, the resultant attraction will be the same as if the mass M were collected at its centre of gravity. Prove also that if M be a segment of a thin exterior confocal ellipsoidal shell, and if its principal axes at its centre of gravity be parallel to the axes of the ellipsoid, the attraction of the ellipsoid on it will reduce to a single force through its centre of gravity. [Math. Tripos.]

Ex. 5. A solid homogeneous ellipsoid is divided into two parts by a plane perpendicular to an axis. Prove that the mutual attraction of the parts for varying positions of the plane varies as the square of the area of section.

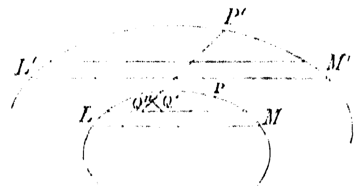
[May Exam. 1881.]

Ex. 6. Show that any plane divides a solid homogeneous ellipsoid into two parts such that the attraction between them reduces to a single force.

[Em. Coll. 1891.]

209. *To find the attraction of a solid homogeneous ellipsoid at an external point P' whose coordinates are ξ', η', ζ'.*

Let R be the distance of any element QQ' of the ellipsoid from P', and let ϕ be the angle this distance makes with the axis of x. In the figure



$$R=QP',\quad \phi=P'QQ'.$$

If $f'(R)$ be the law of attraction, the x component of the attraction of this element at P' is

$$\rho\,dxdydz\,f'(R)\cos\phi.$$

Drawing $Q'n$ perpendicular to $P'Q$, it is obvious that

$$\cos\phi=\frac{Q'n}{QQ'}=-\frac{dR}{dx}.$$

The x attraction of the element at P', measured positively in the positive direction of x, is therefore $\rho\,dydz\,f'(R)\,dR$. Let LM be a column having its length LM parallel to the axis of x and the elementary area $dydz$ for base. Integrating with regard to R we find that the x component of its attraction at P' is

$$\rho\,dydz\int f'(R)\,dR=\rho\,dydz\,\{f(P'M)-f(P'L)\}.$$

Let us now describe an ellipsoid through P' confocal to the external surface of the attracting solid. Let a', b', c' be the semi-

axes of this new ellipsoid. If L', M', P be points corresponding to L, M, P', the column $L'M'$ will have for its base the elementary area $dy'dz'$, where $y'/b' = y/b$ and $z'/c' = z/c$. The coordinates ξ, η, ζ of P are known in terms of those of P' by similar relations; see Art. 188.

The attraction of the column $L'M'$ at P is

$$\rho dy'dz' \{f(PM') - f(PL')\}.$$

By Ivory's theorem, $P'M = PM'$, $P'L = PL'$; the x attractions of the columns $LM, L'M'$ are therefore in the ratio of the areas $dydz$, $dy'dz'$ of their bases, i.e. the x attractions are in the constant ratio bc to $b'c'$.

If we fill one ellipsoid with columns like LM, the other ellipsoid is filled by the corresponding columns, and the x attractions of the corresponding columns are in the same ratio. We therefore infer that

$$\frac{x \text{ att}^{\text{n}} \text{ of inner ellip}^{\text{d}} \text{ at } P'}{x \text{ att}^{\text{n}} \text{ of outer ellip}^{\text{d}} \text{ at } P} = \frac{bc}{b'c'}.$$

Similar theorems apply to the y and z components of the attractions of the two ellipsoids.

210. When the law of attraction is the inverse square, the axial components of the attraction of the outer ellipsoid at the internal point P or (ξ, η, ζ) are

$$X = -A'\xi, \qquad Y = -B'\eta, \qquad Z = -C'\zeta.$$

The axial components of the inner ellipsoid at the external point P' or (ξ', η', ζ') are therefore given by

$$X' = -A'\xi\frac{bc}{b'c'} = -A'\frac{abc}{a'b'c'}\xi',$$

$$Y' = -B'\frac{abc}{a'b'c'}\eta', \quad Z' = -C'\frac{abc}{a'b'c'}\zeta'.$$

Here a', b', c' are the semi-axes of the confocal drawn through the attracted point P', and A', B', C' are the same functions of the ratios of the axes a', b', c' that A, B, C in Art. 207 are of the ratios of a, b, c.

211. From these values of X', Y', Z' we may at once deduce Maclaurin's theorem. If we compare the attractions at the same point of two different ellipsoids bounded by confocals, we notice that a', b', c' are the same for each, so that each of the components X', Y', Z' is proportional to abc, i.e. to the product of the axes. *The attractions* therefore *at the same external point of different ellipsoidal bodies bounded by confocals are the same in direction and their magnitudes are proportional to their masses.*

212. Ex. 1. If a thin layer of attracting matter, distributed over the surface of an ellipsoid, be such that the surface density ρ at any point (x, y, z) is $p(Lx+My+Nz)$, where p is the perpendicular on the tangent plane, prove (1) that the axial components of the attraction at any internal point are constant and respectively equal to La^2A, Mb^2B, Nc^2C, where A, B, C, have the meaning given in Art. 207 and (2) that the potential is a linear function of the coordinates.

To prove this we regard the layer as occupying the space between two indefinitely near ellipsoids. If the ellipsoids are equal, similar and similarly situated, we show that the thickness t is given by $$t=p\left(x\frac{\delta f}{a^2}+y\frac{\delta g}{b^2}+z\frac{\delta h}{c^2}\right),$$
where δf, δg, δh are the coordinates of the centre of one ellipsoid referred to the axes of the other.

Subtracting the axial components of the attraction of one ellipsoid from those of the other, we see, by Art. 207, that the axial components of the attraction of the shell are $X=A\delta f$, $Y=B\delta g$, $Z=C\delta h$. The result easily follows. Since the quantities L, M, N may be multiplied by any factor without altering the truth of the theorem, we notice that L, M, N need not be indefinitely small.

Ex. 2. If a thin layer of attracting matter distributed on the surface of an ellipsoid be such that the surface density at any point (x, y, z) is $pf(x, y, z)$ where f is a homogeneous quadratic function of (x, y, z), prove (1) that the potential at any internal point is also a quadratic function of the coordinates of that point together with a constant, and (2) that the axial components of the attraction at any internal point are linear functions of the coordinates of that point.

To prove this we regard the thin layer as occupying the space between two concentric ellipsoids, having their axes nearly coincident in direction, each with each. Let (a, b, c), $(a+da$, &c.) be the semi-axes. We show that the thickness t is given by $$\frac{t}{p}=x^2\frac{da}{a^3}+\dots+xy\frac{a^2-b^2}{a^2b^2}\cos x'y+\dots,$$
where (x', y', z') are the axes of the outer ellipsoid. Thus by choosing da, db, dc and the cosines $\cos x'y$ &c. properly, this thin layer may be made to represent the given quadratic distribution over the surface.

Subtracting the axial components of one ellipsoid from those of the other, we find that the x component of the attraction of the shell is
$$X=x\left(\frac{dA}{da}da+\frac{dA}{db}db+\frac{dA}{dc}dc\right)+y(A-B)\cos x'y+z(A-C)\cos x'z,$$
with similar expressions for the components parallel to y and z.

Ex. 3. The surface density at any point of an ellipsoid is given by $\rho=kpz^2$: find the potential at any internal point.

Since this is a quadratic function of the coordinates, the potential at any internal point ξ, η, ζ is of the form $V=N+A\xi^2+B\eta^2+C\zeta^2+2D\eta\zeta+2E\zeta\xi+2F\xi\eta$. But the expression must be symmetrical about the coordinate planes, hence $D=0$, $E=0$ and $F=0$. Again V must satisfy Laplace's equation, hence $A+B+C=0$. We therefore have $$V=N+A\xi^2+B\eta^2-(A+B)\zeta^2.$$

If then we find by integration the potential at the centre and any other two convenient points, the potential is known throughout.

Ex. 4. The surface density at any point of an ellipsoid is $\rho=kpxz$: show that the potential at any internal point is given by $V=2Exz$ where E is a constant.

213. **Elliptic cylinders.** *To find the attraction at an internal point of a solid homogeneous cylinder whose cross section is an ellipse and whose length is infinite in both directions.*

The axial components of this attraction may be immediately deduced from those of an ellipsoid by making one of the axes infinite. Let us make $c=\infty$, so that the infinite cylinder stands on an ellipse whose axes are along the axes of x and y. The axial components of the attraction at any internal point (ξ, η, ζ) are

$$X=-A\xi, \qquad Y=-B\eta, \qquad Z=0,$$

where $$A=\rho\int\frac{x^2}{a^2}\,d\omega \text{ and } B=\rho\int\frac{y^2}{b^2}\,d\omega.$$

Since in a cylinder (x, y) may be regarded as the coordinates of any point on the elliptic section, we have obviously

$$A+B=4\pi\rho, \qquad Aa^2+Bb^2=\rho\int r'^2 d\omega,$$

where r' is the radius vector of the cross section in the plane of xy. Putting for $d\omega$ its usual polar value $\sin\theta d\theta d\phi$ we have

$$\int r'^2 d\omega=\int\sin\theta d\theta\,.\int r'^2 d\phi,$$

where the limits are $\theta=0$ to π and $\phi=0$ to 2π. The first integral is obviously equal to 2 and the second integral is twice the area of the ellipse, i.e. $2\pi ab$. We thus have

$$Aa^2+Bb^2=4\pi\rho ab.$$

The axial components are therefore

$$X=-4\pi\rho\frac{ab}{a+b}\frac{\xi}{a}, \qquad Y=-4\pi\rho\frac{ab}{a+b}\frac{\eta}{b}.$$

214. *To find the attraction at an external point of a solid homogeneous elliptic cylinder.*

The attraction of an ellipsoid at an external point has already been deduced from that at an internal point by an application of Ivory's theorem. The same arguments apply to cylinders, and to avoid repetition we take the result from Art. 210.

Since a', b', c' are the semi-axes of a confocal through the attracted point, $$a'^2-a^2=b'^2-b^2=c'^2-c^2.$$

Since a'^2-a^2 is finite, it follows that when c and c' are both infinite, their ratio is unity. The component X' is therefore

$$X'=-A'\frac{ab}{a'b'}\xi'=-4\pi\rho\frac{b'}{a'+b'}\frac{ab}{a'b'}\xi',$$

by substituting for A' its value found in Art. 213.

In this way we find that the axial components X', Y', Z' of the attraction of a solid cylinder at an external point (ξ', η', ζ') are

$$X' = -4\pi\rho \frac{ab}{a'+b'} \frac{\xi'}{a'}, \quad Y' = -4\pi\rho \frac{ab}{a'+b'} \frac{\eta'}{b'}, \quad Z' = 0,$$

where (a', b') are the semi-axes of a cross section of a confocal cylinder drawn through the attracted point.

215. Ex. 1. Show that the resultant attraction of an infinite cylinder is the same in magnitude at all *internal* points situated on a coaxial cylinder similar and similarly situated to the boundary. Show also that the direction of the attraction at any point on the surface of such a cylinder is parallel to the eccentric line of that point.

Ex. 2. Show that the resultant attraction of an infinite cylinder is the same in magnitude at all *external* points situated on a cylinder confocal with the boundary. Show also that its direction at any point on a confocal is parallel to the eccentric line of that point.

Ex. 3. If a thin stratum of attracting matter distributed on the surface of an infinite elliptic cylinder be such that the surface density at any point (x, y, z) is $p\left(L\frac{x}{a} + M\frac{y}{b} + N\right)$, prove that the axial components of the attraction at an internal point (ξ, η, ζ) are

$$X = L\frac{4\pi ab}{a+b}, \qquad Y = M\frac{4\pi ab}{a+b}, \qquad Z = 0,$$

where the coordinate axes are the principal diameters of a cross section and the axis of the cylinder.

Ex. 4. If the surface density ρ of a thin stratum of attracting matter placed on the surface of an infinite elliptic cylinder be given by

$$\rho = p\left(L\frac{x^2}{a^2} + M\frac{xy}{ab} + N\frac{y^2}{b^2}\right),$$

prove that the x component of the attraction at any internal point (ξ, η) is

$$X = \frac{4\pi ab}{(a+b)^2}\{(L-N)x + My\},$$

with a similar expression for the y component.

Ex. 5. Show that the potential at an internal point of an infinite cylindrical mass bounded by two coaxial similar and similarly situated cylinders is infinite.

Ex. 6. The components of the attraction of a right elliptic cylinder whose section is $(x/a)^2 + (y/b)^2 = 1$, and whose ends are any two planes perpendicular to the axis, at an external point ξ', η', ζ', are X', Y', Z'. A confocal cylinder having the same ends is described through ξ', η', ζ', and attracts an internal point ξ, η, ζ, with components X, Y, Z. Show that if $\xi/a = \xi'/a'$, $\eta/b = \eta'/b'$, $\zeta = \zeta'$, then $X'/X = b/b'$, $Y'/Y = a/a'$. [Math. T. 1879.]

216. **Heterogeneous ellipsoidal layer.** *A thin layer of matter is placed on the surface of an ellipsoid, such that the surface density at any point Q is $p\phi(x, y, z)$, where ϕ is an integral rational*

*homogeneous function of the kth degree of the coordinates of Q. It is required to find the potential at any internal point P**.

If $d\sigma$ be any elementary area of the surface at Q, and D the distance of Q from P, the potential of the layer at P is

$$V=\int\frac{\phi(x, y, z)\,pd\sigma}{D}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

the integration extending over the whole surface of the ellipsoid. Let $d\omega$ be the angle subtended at the centre O by the elementary area $d\sigma$. Let $OQ=r$, $OP=R$ and $\cos POQ=q$.

If (l, m, n) be the direction cosines of OQ and (λ, μ, ν) those of OP, we have

$$\frac{1}{r^2}=\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$q=l\lambda+m\mu+n\nu\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Since $pd\sigma$ is three times the volume of the tetrahedron whose base is $d\sigma$ and vertex O, we have $pd\sigma=r^3d\omega$. The potential is therefore

$$V=\int\frac{\phi(x, y, z)\,r^3d\omega}{(r^2+R^2-2rRq)^{\frac{1}{2}}}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The attracted point being within the surface we expand the denominator in a series of ascending powers of R/r. We thus have

$$V=\int\phi(x, y, z)\,r^2d\omega\left(1+P_1\frac{R}{r}+P_2\frac{R^2}{r^2}+\ldots\right)\ldots\ldots(5),$$

where P_1, P_2, &c. are commonly called Legendre's functions of q.

* The potentials of heterogeneous solid ellipsoids and ellipsoidal shells have been discussed in several ways. First there is the important paper of Green read to the Cambridge Philosophical Society in 1833, in which the subject is treated in a very general manner; the law of attraction being the inverse nth power and the density is of the form $u^nf(x, y, z)$, where f is an integral function and u represents $1-x^2/a^2-y^2/b^2-z^2/c^2$. Green uses Cartesian coordinates throughout, but a solution has also been obtained by the use of elliptic coordinates and Lamé's functions. An account of this method may be found in the treatise on *Spherical Harmonics* by Dr Ferrers, now Master of Caius College. The reader may also consult a paper on the potentials of ellipsoids &c. by the Master of Caius College in vol. XIV. of the *Quarterly Journal*, 1877. The method adopted is first to investigate the potential of an ellipsoid whose density is any function $f(u)$, where u has the same meaning as before. Next it is shown that if the density be zero at the boundary the differential coefficient of this potential with regard to any variable, say x, is the potential of an ellipsoid whose density is $df(u)/dx$. If this density is again zero at the boundary, this process may be repeated. It is then pointed out that $x^\alpha y^\beta z^\gamma$ may be expressed by means of differential coefficients of powers of u. Thus the potential of a solid ellipsoid whose density is any integral rational function of x, y, z is determined. The potentials of shells, laminæ and rings are then deduced.

Another method by Mr Dyson is given in vol. XXV. of the *Quarterly Journal*, 1891. In the first part of the paper the potential of an ellipsoidal shell is found by a method nearly the same as that adopted in the text. The results thus obtained suggest the general form of expansions for the internal and external potential when the density is kpx^n or $kpf(x, y, z)$. These assumed forms are shown by actual substitution to satisfy Laplace's equation. Finally the law of density of the stratum is connected with these expansions by using Green's theorem, see Art. 117. The paper terminates with the potentials of solid ellipsoids of variable densities.

Putting $\phi = r^k\psi_k$, where ψ_k is a homogeneous function of l, m, n of the kth degree, we may write the expansion (5) in the form

$$V = \int d\omega r^{k+2}\psi_k \left\{1 + P_1 \frac{R}{r} + \&c. + P_k \left(\frac{R}{r}\right)^k\right\}$$

$$+ R^{k+1}\int d\omega r\psi_k \left\{P_{k+1} + P_{k+3}\left(\frac{R}{r}\right)^2 + \&c.\right\}$$

$$+ R^{k+2}\int d\omega\psi_k \left\{P_{k+2} + P_{k+4}\left(\frac{R}{r}\right)^2 + \&c.\right\} \text{..........(6)}.$$

Now $r^{h+k}P_h\psi_k$ is a homogeneous function of the Cartesian coordinates of Q of $h+k$ dimensions; hence since the ellipsoid is symmetrical about the coordinate planes

$$\int r^f . r^{h+k}P_h\psi_k\, d\omega = 0,$$

where $h+k$ is an odd integer and f is any positive or negative integer. It is therefore evident that the alternate terms in the first line of (6) are zero, and also that every term in the second line is zero.

The terms in the third line are included in the general form $R^{k+\alpha}\int d\omega\, \dfrac{\psi_k P_{k+\alpha}}{r^{\alpha-2}}$. Since α is here an even integer greater than zero, we can substitute for $1/r^{\alpha-2}$ its value given by (2) without introducing any square roots. After the substitution $\psi_k/r^{\alpha-2}$ becomes a homogeneous function of l, m, n of a degree less than that indicated by the suffix of $P_{k+\alpha}$. When this quotient is expressed in a finite series of Laplace's functions, not one will rise to the order indicated by the suffix of $P_{k+\alpha}$. Hence by the first property of Laplace's functions mentioned in Art. 167, the result of the integration is zero.

In this way the expression for V is reduced to the alternate terms in the first line. Writing these in the reverse order, we have

$$V = \int d\omega r^{k+2}\psi_k \left\{P_k\left(\frac{R}{r}\right)^k + P_{k-2}\left(\frac{R}{r}\right)^{k-2} + \dots\right\} \text{......(7)},$$

where the integrations extend over the surface of the ellipsoid.

217. Since P_k may obviously be written in the form of a homogeneous function of λ, μ, ν of the kth degree (as well as of l, m, n), it follows that R^kP_k is a homogeneous function of the coordinates (ξ, η, ζ) of the attracted particle. The potential due to the distribution of density represented by $\rho = pr^k\psi_k$ has therefore been expressed in the form $V = u_k + u_{k-2} + u_{k-4} + \&c. \dots (8)$,

where u_k, u_{k-2} &c. are homogeneous functions of the coordinates of the attracted particle respectively of the kth, $(k-2)$th degrees.

We infer the following theorem, *if the surface density ρ at any point Q be such that ρ/p is an integral rational homogeneous function of k dimensions of the coordinates of Q, the potential at any internal point P is the sum of a series of integral rational homogeneous functions of the coordinates of P of the degrees k, $k-2$, &c. respectively.*

218. The several terms in the expression for V are double integrals, but each can be integrated once in the manner explained in Art. 187. In this way the potential of a thin heterogeneous ellipsoidal stratum is reduced to single integrals.

To effect the transformation we notice that P_h, as well as ψ_k, can be expressed as an integral rational function of l, m, n and that $h+k$ is an even integer. Thus each term of the expression for V given in (7) is the sum of a number of integrals of the form $\int r^h l^{2i} m^{2j} n^{2k} d\omega$, the integrations being taken over the whole ellipsoid. There are also terms with odd powers of l, m, n, but since the ellipsoid is symmetrical about its coordinate planes, each of these is zero. We also notice that the highest power of r which occurs does not exceed the sum of the powers of l, m, n by more than 2.

219. To effect the integrations, we take the expression for $\int r^2 d\omega$ found in Art. 185 after integration with regard to ϕ. Introducing the factor $(\cos\theta)^{2p}$, which is constant during the integration with regard to ϕ, and remembering that $u = c\tan\phi$, we have
$$\int r^2 (\cos\theta)^{2p} d\omega = 2\pi abc \int \frac{c^{2p}\, du^2}{(a^2+u^2)^{\frac{1}{2}}(b^2+u^2)^{\frac{1}{2}}(c^2+u^2)^{p+\frac{1}{2}}},$$
where the limits on the right-hand side are zero and infinity.

Putting $a^2 = 1/\alpha$, $b^2 = 1/\beta$, $c^2 = 1/\gamma$ and $u^2 = 1/v$ this becomes, after substituting for r its value given by (2) Art. 216,
$$\int \frac{n^{2p} d\omega}{\alpha l^2 + \beta m^2 + \gamma n^2} = 2\pi \int \frac{v^{p-\frac{1}{2}} dv}{(\alpha+v)^{\frac{1}{2}}(\beta+v)^{\frac{1}{2}}(\gamma+v)^{p+\frac{1}{2}}}.$$

Differentiating this equation i times with regard to α, j times with regard to β, we have
$$\int \frac{l^{2i} m^{2j} n^{2p} d\omega}{(\alpha l^2 + \beta m^2 + \gamma n^2)^{i+j+1}} = H_1 \cdot \int \frac{v^{p-\frac{1}{2}} dv}{(\alpha+v)^{i+\frac{1}{2}}(\beta+v)^{j+\frac{1}{2}}(\gamma+v)^{p+\frac{1}{2}}},$$
where
$$H_1 = 2\pi \frac{1\,.\,3\,.\,5\ldots(2i-1)\,.\,1\,.\,3\,.\,5\ldots(2j-1)}{1\,.\,2\,.\,3\ldots(i+j)\,2^{i+j}}.$$

Comparing these integrals we notice that $1\,.\,3\ldots(2i-1)$ is to be interpreted as equal to unity when i is zero.

Here p is any positive or negative integer. If k be greater than p, differentiate

this $k-p$ times with regard to γ, but if k be less than p integrate it $p-k$ times with regard to γ from $\gamma=\infty$ to $\gamma=\gamma$. We then have

$$\int \frac{l^{2i}m^{2j}n^{2k}d\omega}{(\alpha l^2+\beta m^2+\gamma n^2)^{i+j+k+1-p}} = H \cdot \int \frac{v^{p-\frac{1}{2}}dv}{(\alpha+v)^{i+\frac{1}{2}}(\beta+v)^{j+\frac{1}{2}}(\gamma+v)^{k+\frac{1}{2}}},$$

where $$H=\frac{2\pi \cdot 1 \cdot 3 \cdot 5 \ldots (2i-1) \cdot 1 \cdot 3 \cdot 5 \ldots (2j-1) \cdot 1 \cdot 3 \cdot 5 \ldots (2k-1)}{2^{i+j+k-p} \cdot 1 \cdot 2 \cdot 3 \ldots (i+j+k-p)\, 1 \cdot 3 \cdot 5 \ldots (2p-1)},$$

provided p is equal to or less than $i+j+k$. If p be greater than $i+j+k$ the denominator in the integral on the left-hand side is either unity or rises into the numerator with a positive exponent. The integration can then be effected by ordinary methods, and the reduction to a single integral becomes unnecessary.

This result may also be written in the form

$$\int \frac{l^{2i}m^{2j}n^{2k}d\omega}{(\alpha l^2+\beta m^2+\gamma n^2)^{i+j+k+1-p}} = N\left(\frac{d}{d\alpha}\right)^i \left(\frac{d}{d\beta}\right)^j \left(\frac{d}{d\gamma}\right)^k \int \frac{v^{p-\frac{1}{2}}dv}{Q},$$

where $$N=(-1)^{i+j+k}\frac{2\pi \cdot 2^p}{1 \cdot 3 \cdot 5 \ldots (2p-1)}\, \frac{1}{1 \cdot 2 \cdot 3 \ldots (i+j+k-p)},$$

and $$Q^2=(\alpha+v)(\beta+v)(\gamma+v).$$

Substituting back r for its value given by (2) Art. 216 the integral on the left-hand side becomes $\int r^h l^{2i}m^{2j}n^{2k}d\omega$, where $h=2(i+j+k+1-p)$.

220. Ex. 1. If $f(l^2, m^2, n^2)$ be a homogeneous function of l^2, m^2, n^2 of s dimensions, show that $$\int r^q f(l^2, m^2, n^2)\, d\omega = Nf\left(\frac{d}{d\alpha}, \frac{d}{d\beta}, \frac{d}{d\gamma}\right)\int_0^\infty \frac{v^{p-\frac{1}{2}}dv}{Q},$$

where $$N=\frac{2\pi \cdot 2^p}{1 \cdot 3 \cdot 5 \ldots (2p-1)}\, \frac{(-1)^s}{1 \cdot 2 \cdot 3 \ldots (s-p)}.$$

Show also that $$\int \frac{f(x, y, z)\, d\omega}{r^{p-1}} = Nf\left(\frac{d}{d\alpha}, \frac{d}{d\beta}, \frac{d}{d\gamma}\right)\int \frac{v^{p-\frac{1}{2}}dv}{Q},$$

where $q=2(s+1-p)$.

Ex. 2. If χ be the angle which a variable central radius vector r of an ellipsoid makes with a fixed straight line OP, whose direction cosines are λ, μ, ν, prove that

$$\int (\cos\chi)^{2s} r^{2q} d\omega = K \cdot \int \left(\frac{\lambda^2}{\alpha+v}+\frac{\mu^2}{\beta+v}+\frac{\nu^2}{\gamma+v}\right)^s \frac{v^{s-q+\frac{1}{2}}dv}{Q},$$

where $$K=\frac{L(2s)}{L(s)}\, \frac{L(s-q)}{L(2s-2q+1)}\, \frac{2\pi}{L(q-1)}\, \frac{1}{2^{2q-1}},$$

$$Q^2=(\alpha+v)(\beta+v)(\gamma+v), \qquad L(x)=1 \cdot 2 \cdot 3 \ldots x.$$

By differentiating this expression with regard to α, β, γ successively, we can find an expression for $$\int l^{2i}m^{2j}n^{2k}(\cos\chi)^{2s}r^{2h}d\omega$$
in terms of single integrals, where h has been written for $l+j+k+q$.

Since P_k is a function of the cosine of the angle χ which the radius vector to any point of the ellipsoid makes with a fixed radius vector, viz. that to the attracted point, this example enables us to find the several terms in equation (7) without reducing them to powers of l, m, n.

221. *To find the potential of the same ellipsoidal layer at an external point P' whose coordinates are (ξ', η', ζ').*

To effect this we use the theorem of Art. 192. The potential of the given shell at the external point P' is the same as that of a

confocal shell drawn through P' at an internal point P whose coordinates are $\frac{a}{a'}\xi', \frac{b}{b'}\eta', \frac{c}{c'}\zeta'$.

It is also necessary that the surface density ρ' of the confocal shell should be such that the masses of corresponding elements are equal. It will be found convenient to express the surface density ρ of the given layer in the form $\rho = p\phi\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$, where ϕ may be a function of the constants a, b, c as well as of x/a, &c. The surface density ρ' of the confocal must be such that $\rho d\sigma = \rho' d\sigma'$. Since $\frac{pd\sigma}{p'd\sigma'} = \frac{abc}{a'b'c}$ by Art. 188, we see that $\rho' = \frac{abc}{a'b'c'}p'\phi\left(\frac{x'}{a'}, \frac{y'}{b'}, \frac{z'}{c'}\right)$.

The potential of this confocal at the internal point P may then be found by equation (11), and this result is the required potential of the given shell at the external point P'.

222. Ex. 1. The surface density of an ellipsoidal layer at any point (x, y, z) is given by $\rho = \beta pxy$; find the potential at any point (ξ, η, ζ).

The potential at any internal point is given by $V = \int d\omega r^4 \psi_2 \left\{P_2\left(\frac{R}{r}\right)^2 + P_0\right\}$, where $\psi_2 = \beta lm$, $P_0 = 1$ and $P_2 = \frac{1}{2}(3q^2 - 1) = \frac{1}{2}\{3(l\lambda + m\mu + n\nu)^2 - 1\}$.

Substituting and omitting all those terms (as already explained) which contain odd powers of l, m, n, we have $V = 3\beta R^2\lambda\mu\int r^2 l^2 m^2 d\omega$.

The integral is equal to $\frac{2\pi}{3}\int \frac{v^{\frac{3}{2}} dv}{(\alpha+v)^{\frac{3}{2}}(\beta+v)^{\frac{3}{2}}(\gamma+v)^{\frac{1}{2}}}$.

The potential V' at an external point $(\xi'\eta'\zeta')$ is the same as the potential of a confocal shell passing through that point at the internal point $a\xi'/a'$, &c. Since the surface density of the confocal shell must be $\rho' = \frac{abc}{a'b'c'}p'\frac{x'y'}{a'b'}\beta ab$, we see that

$$V' = 3\beta\frac{abc}{a'b'c'}\left(\frac{ab}{a'b'}\right)^2 \xi'\eta'\int r'^2 l^2 m^2 d\omega,$$

where r' is that radius vector of the confocal whose direction cosines are l, m, n.

Ex. 2. The surface density of an ellipsoidal layer at any point x, y, z is given by $\rho = \beta px^2$; show that the potential at any internal point is $V = E\xi^2 + F\eta^2 + G\zeta^2 + H$, where $E = \frac{1}{2}\beta\int(3l^2 - 1)x^2 d\omega$, $H = B\int l^2 r^4 d\omega$, and F, G may be obtained from E by writing m and n respectively for l.

223. **The inverse fourth power.** Prof. Townsend has given some interesting theorems on the attraction of an ellipsoid when the law is as the inverse fourth power of the distance. A few of these are shown in the following examples; we refer the reader to his paper in the *Quarterly Journal*, Vol. XII.

The attraction for the inverse sixth and higher even powers may be deduced from these by the principle mentioned in Art. 77.

If the law of attraction on any particle be mass divided by the fourth power of the distance, prove that the attraction of a solid homogeneous ellipsoid at an internal point is normal to the similar and similarly situated ellipsoid through that point, and that its magnitude varies inversely as the perpendicular from the point attracted on its polar plane with respect to the bounding surface of the mass.

Prove that the attraction of a solid ellipsoid at an external point is normal to the confocal ellipsoid through that point, and that for points situated on the same confocal the magnitude varies directly as the perpendicular from the centre on the tangent plane to the confocal.

If the law of attraction be the inverse fourth power, show that the potential of a thin shell bounded by ellipsoids (abc), $(a+da,\ b+db,\ c+dc)$ at any internal point is $\frac{4}{3}\pi\frac{1}{1-n^2}\left(\frac{x^2}{a^3}da+\frac{y^2}{b^3}db+\frac{z^2}{c^3}dc\right)+\frac{1}{3}\left(\frac{A}{a}da+\frac{B}{b}db+\frac{C}{c}dc\right)$, where $n^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$, and A, B, C are the three integrals used when the attraction follows the law of nature, viz. $A=\int\frac{x^2}{a^2}d\omega$, $B=\int\frac{y^2}{b^2}d\omega$, $C=\int\frac{z^2}{c^2}d\omega$.

When the surfaces are similar, show that the potential is $\frac{4}{3}\pi\frac{1}{1-n^2}\frac{da}{a}$.

Show that the potential of a solid ellipsoid at an external point is with the usual notation $\frac{4}{3}\pi abc\int\frac{\frac{1}{2}d\lambda}{\lambda a'b'c'}$.

If the law of attraction be the inverse fourth power of the distance, show that the attractions of a solid ellipsoid at an internal and external point are respectively

$$X=\frac{4}{3}\pi\frac{1}{1-n^2}\frac{x}{a^2},\quad Y=\&c.,\quad Z=\&c.,\qquad X'=\frac{4}{3}\pi\frac{abc}{a'b'c'}\frac{1}{\lambda}\frac{p'^2x'}{a'^2},\quad Y'=\&c.,\quad Z'=\&c.,$$

where $n^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$, (a', b', c') are the semi-axes of a confocal through (x', y', z') the external attracted point, and p' is the length of the perpendicular from the centre on the tangent plane to that confocal.

Find also the direction and magnitude of the resultant force in each case.

Potentials of rectilinear figures.

224. **Potential of a lamina.** *To find the potential at any point P of a plane lamina of unit surface density.*

Let PN be the perpendicular from P on the plane. Let the plane of the lamina be the plane of xy, N the origin and NP the axis of z. Let $NP=\zeta$. Let (r, θ) be the polar coordinates of a point on the plane of xy.

If QQ' be any elementary arc of the curvilinear boundary, the potential of the triangular area NQQ' is $\int\frac{r\,d\theta\,dr}{(r^2+\zeta^2)^{\frac{1}{2}}}$, where the limits of integration are $r=0$ and $r=r$. If $R=PQ$, this reduces to $(R-\zeta)\,d\theta$.

Integrating this again for all the elements of the boundary, we see that *the potential V' at P of the area of any closed plane curve* is $\int(R-\zeta)\,d\theta$. In this expression the limits are determined by making the point Q (whose coordinates are (r, θ)) travel completely round the curve in the positive direction, the elementary angle $d\theta$ having its proper sign according as the radial angle θ is increasing or decreasing when Q passes over each element of the perimeter.

When the perpendicular PN falls within the lamina, the limits of θ are 0 and 2π, the expression for the potential is then $\int R\,d\theta-2\pi\zeta$. When the perpendicular falls outside the lamina the upper and lower limits of θ are the same, so that $\int\zeta\,d\theta=0$ and the expression for the potential is simply $\int R\,d\theta$.

225. We may put the expression just found for the potential into another form which is sometimes more useful.

If $rd\theta dr$ is any element of the area of the triangle NQQ', u its distance from P and ϕ the angle u makes with the normal to the plane, the solid angle $d\omega$ subtended at P by the triangle is

$$d\omega = \int \frac{rd\theta dr}{u^2} \cos\phi = \int \frac{\zeta du d\theta}{u^2} = \left(1 - \frac{\zeta}{R}\right) d\theta,$$

the limits of u being ζ and R.

The potential of the triangular area NQQ' at P is equal to

$$\frac{R^2 d\theta}{R} - \zeta d\theta = \frac{r^2 d\theta}{R} - \zeta d\theta\left(1 - \frac{\zeta}{R}\right) = \frac{r^2 d\theta}{R} - \zeta d\omega.$$

In fig. 1, the perpendicular PN falls within the attracting area. We then find, by integrating all round the perimeter of the area, that the potential at P is

$$V' = \int \frac{r^2 d\theta}{R} - \zeta\omega,$$

where ω is the solid angle subtended at P by the area.

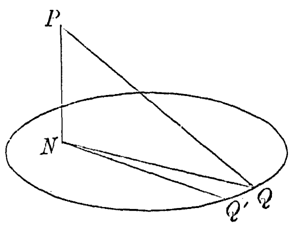


Fig. 1.

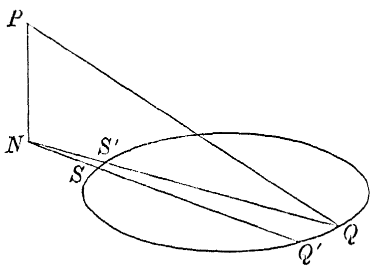


Fig. 2.

In fig. 2, the perpendicular PN falls without the area. In this case we must subtract from the potential of NQQ' that of NSS'. Since $d\theta$ is positive for QQ' and negative for $S'S$ when a point travels round the curve in the positive direction, the form of the result is unaltered.

Let ds be the length of any elementary arc QQ' of the perimeter, p the perpendicular from N on the tangent at Q. Then since $r^2 d\theta = pds$, the potential at P of the area takes the form $V' = \int \frac{pds}{R} - \zeta\omega$, where the integration extends all round the perimeter, and ω is the solid angle subtended by the lamina at P.

Ex. 1. If the law of force be the inverse fifth power of the distance, show that the potential of a plane lamina of unit density at a point P is $\frac{1}{8\zeta^2}\int \frac{pds}{R^2}$, where the integration extends all round the perimeter and the letters have the same meaning as in Art. 224.

226. When the lamina is bounded by rectilinear sides, p is constant for each side and may therefore be brought outside the integral sign. The integral $\int ds/R$ is then the potential of that side at P. We therefore have the following theorem.

If V' be the potential at any point P of the area contained by any plane rectilinear figure regarded as of unit surface density; V_1, V_2, &c. the potentials at the same point of its sides each regarded as of unit line density, ω the solid angle subtended at P by the area, then $V' = -\zeta\omega + p_1V_1 + p_2V_2 + \&c.$, where ζ is the length of the perpendicular PN on the area, and p_1, p_2, &c. are the perpendiculars from N on the sides taken with their proper signs.

The signs of the perpendiculars are determined by the following rule. If the point Q travel round the perimeter in the direction of the motion of the hands of a watch, the perpendicular p is positive or negative according as the origin N lies on the right or left hand side of the tangent at Q.

227. **Potential of a solid.** *If V'' be the potential at any point P of a solid, of unit density, and bounded by plane rectilinear faces; V_1', V_2', &c. the potentials at the same point of its faces each regarded as of unit surface density, then*

$$2V''=\zeta_1 V_1+\zeta_2 V_2+\ldots,$$

where ζ_1, ζ_2, &c. are the perpendiculars from P on the faces taken with their proper signs.

Describe an elementary cone whose vertex is P and whose base is any element of area of the boundary of the solid. Let $d\omega$ be its solid angle. The volume of an element of the cone being $r^2 d\omega dr$, the potential of the cone at P is

$$\int\frac{r^2 d\omega dr}{r}=\tfrac{1}{2}r^2 d\omega=\tfrac{1}{2}\frac{pd\sigma}{r},$$

where r is now the radius vector drawn from P to the elementary area $d\sigma$ and p is the perpendicular from P on the tangent plane. The potential of the whole solid body at P is therefore $\frac{1}{2}\int\frac{pd\sigma}{r}$.

When the boundaries of the solid are planes, p is constant for each plane and $\int pd\sigma/r$ is the potential of that plane face at P. We have at once $V''=\frac{1}{2}\Sigma pV'$.

228. The solid angle subtended at any point P by any triangle ABC is the area of the unit sphere enclosed by the planes PAB, PBC, PCA. This area is the same as that of the spherical triangle traced on the sphere by these planes, and a finite expression for its value is given in books on spherical trigonometry. Since any polygonal area can be divided into triangles it follows that the solid angle subtended at P by any rectilinear figure can always be found. The result may be complicated but it involves no integrations which cannot be effected.

It immediately follows from Arts. 226, 227 that *the potentials of all rectilinear figures and the potentials of all solids bounded by plane rectilinear faces can be found. Thus the three integrals which express the components of the attraction of a rectilinear lamina or solid can be found in finite terms.*

229. **Components of Attraction.** Some simple expressions may be found for the components of the attraction of the lamina. We know by Playfair's theorem, that the component along the perpendicular PN on the lamina is equal to the solid angle subtended at P by the lamina, see Art. 26.

By using Gauss' theorem (Art. 16) we may obtain an expression for *the resolved part of the attraction along a straight line drawn in the plane.* If this straight line be called the axis of x and the boundary of the lamina be a closed curve in the plane of xy, the x component of the attraction is

$$X'=\int\frac{dy}{R},$$

where R is the distance of an element of the boundary from the attracted point P.

To prove this we proceed as in Art. 16. We divide the lamina into elementary rectangles having their length parallel to the axis of x, and whose breadth is dy. If AB be any one of these (regarded as of unit surface density), its x attraction on P in the direction AB is $\left(\frac{1}{PA}-\frac{1}{PB}\right)dy$, see Art. 11. The attraction of the whole

lamina is therefore $\int \frac{dy}{R}$, where R stands for either PA or PB, and dy is taken positive or negative according as the ordinate y is increasing or decreasing when a point Q travelling round the curve passes A or B.

230. *A solid body of unit density is bounded by plane faces: it is required to find the resolved part of its attraction at a given point P in a given direction Px.*

This is a simple corollary from Gauss' theorem, Art. 16. Whatever the form of the solid may be, its component of attraction in the direction Px is $X'' = \int \frac{d\sigma \cos\phi}{R}$, where $d\sigma$ is an element of the surface, ϕ the angle the normal at $d\sigma$ makes with the given direction Px and R is the distance of $d\sigma$ from P.

When the solid is bounded by plane faces, $\cos\phi$ is the same for all the elements of the same face. It may therefore be brought outside the integral sign. Since the integral $\int d\sigma/R$ is obviously the potential at P of the face, we have at once

$$X'' = V_1' \cos\phi_1 + V_2' \cos\phi_2 + \ldots\ldots = \Sigma V \cos\phi,$$

where V_1', V_2', &c. are the potentials at P of the plane faces regarded as of unit surface density, and ϕ_1, ϕ_2, &c. are the angles the normals measured inwards make with the direction in which X is measured.

231. Ex. 1. If α, β, γ, δ be the quadriplanar coordinates of a point P referred to the faces of a tetrahedron, show that the potential of the solid contained by the tetrahedron regarded as of unit density is $\frac{1}{2}(V_1\alpha + V_2\beta + V_3\gamma + V_4\delta)$, where V_1, V_2, V_3, V_4 are the potentials at the same point of the several faces regarded as of unit surface density.

Ex. 2. Show that the solid angle ω subtended at any point P by a triangular area ABC is given by

$$\left(\tfrac{3}{2}v \operatorname{cosec}\frac{\omega}{2}\right)^2 = \frac{(q+r)^2 - a^2}{4} \cdot \frac{(r+p)^2 - b^2}{4} \cdot \frac{(p+q)^2 - c^2}{4},$$

where v is the volume of the tetrahedron $ABCP$ and p, q, r are the distances of P from the angular points of the triangle.

Ex. 3. The triangle OBC is right-angled at B and at O a straight line OP is drawn perpendicular to its plane. If the triangle be of unit surface density, prove that its attractions at P resolved parallel to OP, OB, and BC respectively are

$$\tan^{-1}\frac{b}{ac}(a^2+b^2+c^2)^{\frac{1}{2}} - \tan^{-1}\frac{b}{c}$$

$$\frac{c}{(b^2+c^2)^{\frac{1}{2}}} \log \frac{(b^2+c^2)^{\frac{1}{2}} + (a^2+b^2+c^2)^{\frac{1}{2}}}{a} - \log \frac{c + (a^2+b^2+c^2)^{\frac{1}{2}}}{(a^2+b^2)^{\frac{1}{2}}}$$

$$\log \frac{b + (a^2+b^2)^{\frac{1}{2}}}{a} - \frac{b}{(b^2+c^2)^{\frac{1}{2}}} \log \frac{(b^2+c^2)^{\frac{1}{2}} + (a^2+b^2+c^2)^{\frac{1}{2}}}{a},$$

where $a = OP$, $b = OB$, $c = BC$. Since any rectilinear figure in the plane of xy may be divided into right-angled triangles having a common corner O by dropping perpendiculars from O on the sides and joining O to the corners, these results give the three resolved attractions of any plane rectilinear figure. [Knight's problem. Todhunter, p. 474.]

Ex. 4. Deduce the expressions for X' and X'' given in Arts. 229, 230 from the values of the potential given in Art. 227.

THE BENDING OF RODS.

Introductory Remarks.

1. Our object in this chapter is to discuss the stretching, bending, and torsion of a thin rod or wire. We may define a rod as a body whose boundary is a tubular surface, of small section. The surface is therefore generated by the motion of a small plane area whose centre of gravity describes a certain curve and whose plane is always normal to the curve. The curve is generally called *the central axis* or *central line* of the rod.

The rod or wire is to be so thin, that, *so far as the geometry of the figure is concerned*, it may be regarded as a curved line having a tangent and an osculating plane. Although this limitation will be generally assumed it will be seen in the sequel that some of the theorems apply to rods of considerable thickness. It is not proposed to enter into the general theory of the elasticity of solid bodies, except where it is necessary for the elucidation of the point under discussion, and even then the reference will be restricted as far as possible to the most elementary considerations.

2. In general the deformation of the body will be regarded as very small, so that each element of the body is only slightly strained from its natural shape. It will therefore be assumed that the whole effect, when properly measured, of any number of disturbing causes may be obtained by superimposing their separate effects.

3. By reference to Art. 142 of the first volume of this treatise, it will be seen that the action across any section C of a thin rod AB consists of a force and a couple. On this is founded the mathematical distinction between a string and a rod. The action across any section of the former is a force, called its tension, which acts along the tangent to the string, Vol. I., Art. 442. In the case of a rod the force may act at any angle to the tangent and there is in addition a couple.

4. Let P be any point of a body, let a closed plane curve be described round P of indefinitely small area, and let this area be ω. If the body is a fluid it is the fundamental principle of hydrostatics that the action between the fluid on one side and the fluid on the other side of the area ω consists of a force whose direction is perpendicular to the plane of the area. It is thence deduced that the magnitude of this force or pressure is the same for all inclinations to the horizon of the elementary curve provided its area remains unaltered. If the body is an elastic solid, the action across the plane is also a force, but its direction is not necessarily perpendicular to the plane of the area and its magnitude is not necessarily the same for all inclinations of the plane.

In discussing the mechanics of a rod, its cross section, though very small, is not to be regarded as infinitely small. If we divide any section into elementary areas, the action across each element will be an elementary force, and the resultant of all these will be, in general, a force and a couple, Vol. I., Art. 143.

The Stretching of Rods.

5. *To determine the simple stretching of a straight rod by a force applied at one extremity, the other being held fast.*

The relation which exists between the force and the extension of the rod has already been discussed in the first volume of this treatise under the name of Hooke's law. If l_1, l be the unstretched and stretched lengths of the rod, ω the area of the section of the unstretched rod, $T\omega$ the tension, then $\dfrac{l-l_1}{l_1}=\dfrac{T}{E}$, where E is a constant depending on the material of the rod and is usually called Young's modulus.

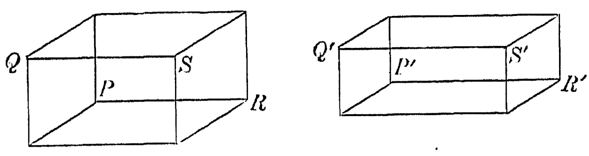
When a rod is stretched we know by common experience that its breadth and thickness are also altered. These lateral changes follow a law similar to Hooke's law except that the modulus E is not necessarily the same as that for extension. The study of these lateral contractions belongs properly to the theory of elasticity and only a simple case will be considered here.

6. The substance of a homogeneous body is called *isotropic* when the properties of a solid of *any* given form and dimensions cut from it are the same whatever directions its sides may have in the body. The substance is called *æolotropic* when the properties of the solid depend on the directions which its sides have in the body. We shall suppose that the material of which the rod is composed is isotropic.

7. **Theory of a stretched rod.** Let the unstretched rod form a cylinder with a cross section of any form and size. When stretched the rod becomes thinner, so that the several particles undergo lateral as well as longitudinal displacements. There is one fibre or line of particles which is undisturbed by the lateral contraction. Let this straight line, which we may regard as the central line, be taken as the axis of x, and let the origin be at the fixed extremity of the rod. We suppose that the stretching forces at the two ends are distributed over the extreme cross sections in such a manner that after the rod is stretched these sections continue to be plane and perpendicular to the central axis. It will appear from the result that the force at each end should be equally distributed over the area.

Let x, y, z be the coordinates of any particle P in the unstrained solid, $x+u$, $y+v$, $z+w$ the coordinates of the same particle P' of matter in the deformed body. Then u, v, w are such functions of x, y, z that the equations of equilibrium of all the elements of the solid are satisfied. We shall now prove that if we take $u=Ax$, $v=-By$, $w=-Bz$ all the equations of equilibrium may be satisfied by properly choosing the constants A and B. According to this supposition the external boundary of the stretched rod will be a cylinder and the particles of matter which occupy any normal cross section of the unstrained rod will continue to lie in a plane perpendicular to the axis when the rod is stretched.

Let $PQRS$ be any rectangular element of the unstrained solid having the faces PQ and RS perpendicular to the central axis. By the given conditions of the question this element assumes in the strained solid a form $P'Q'R'S'$ in which all the angles are still right angles and the sides parallel to their original directions. The direction of the stress across each face of the strained element is therefore perpendicular to that face. To measure these forces we refer each to a unit of area. Let N_x, N_y, N_z be the forces, so referred; let these act on the three faces which meet



at the corner P' and are respectively perpendicular to the axes of x, y, z; we shall regard these forces as positive when (like the tension of a string) they pull the matter on which they act, and as negative when (like a fluid pressure) they push.

Let a, b, c and $a(1+\alpha)$, $b(1+\beta)$, $c(1+\gamma)$ be the sides of the element before and after the deformation. Then N_x, N_y, N_z are functions of α, β, γ, see Art. 489, Vol. I. We shall expand these functions in ascending powers of α, β, γ and since we here confine our attention to a first approximation, we shall neglect all the higher powers of α, β, γ. Assuming the lowest powers in the expansion to be the first, we have
$$N_x=\kappa\alpha+\lambda(\beta+\gamma),$$
the coefficients of β and γ being the same because the medium is isotropic. For the same reason the stress N_y must be the same function of β and γ, α, that N_x is of α and β, γ. Thus
$$N_y=\kappa\beta+\lambda(\alpha+\gamma).$$
In the same way N_z may be derived from N_x by interchanging α and γ. To make these more symmetrical, it is usual to write them in the form
$$N_x=2\mu\alpha+\lambda(\alpha+\beta+\gamma),\qquad N_y=2\mu\beta+\lambda(\alpha+\beta+\gamma),\qquad N_z=2\mu\gamma+\lambda(\alpha+\beta+\gamma).$$
The constants λ and μ are the same as those chosen by Lamé to measure the

elastic properties of a solid; see his *Leçons sur la théorie mathématique de l'élasticité des corps solides.*

8. In the problem under consideration the sides dx, dy, dz of the unstrained element become $dx+du$, $dy+dv$, $dz+dw$. It follows that

$$\alpha=\frac{du}{dx}, \quad \beta=\frac{dv}{dy}, \quad \gamma=\frac{dw}{dz}.$$

Substituting the assumed values of u, v, w, we have

$$N_x=2\mu A+\lambda(A-2B), \qquad N_y=-2\mu B+\lambda(A-2B), \qquad N_z=-2\mu B+\lambda(A-2B).$$

These values are independent of x, y, z, so that the opposite faces of any element *wholly internal* are acted on by equal and opposite forces. It follows that every internal element is in equilibrium.

Consider next the elements which have one or more of their faces on the boundary of the rod. Such faces must be parallel to the central axis and in a vacuum are not acted on by any pressure. It is therefore necessary for their equilibrium that the constant forces represented by N_y and N_z should be zero. We therefore have $$\frac{B}{A}=\frac{\lambda}{2(\lambda+\mu)}, \qquad N_x=\frac{(3\lambda+2\mu)\mu}{\lambda+\mu}A.$$

Since Ax is the extension, By the contraction of a rod of length x and breadth y and N_x is the stretching force per unit of area of the section, it follows that

$$\frac{\text{increase of length}}{\text{original length}}=\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}N_x,$$

$$\frac{\text{decrease of breadth}}{\text{original breadth}}=\frac{\lambda}{2\mu(3\lambda+2\mu)}N_x.$$

Comparing the first of these with the statement of Hooke's law given in Vol. I. Art. 489, we see that the constant E, usually called Young's modulus, is the reciprocal of the coefficient of N_x. If E' be the corresponding coefficient for the decrease of breadth we have $$E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \qquad E'=\frac{2(\lambda+\mu)}{\lambda}E.$$

It follows from this solution that when a rod has been stretched, each fibre (or column of particles parallel to the central axis) is stretched and contracted independently of the others and exerts no action on the neighbouring fibres. The total force required to produce a given extension is therefore independent of the form of the cross section provided its area remains unaltered.

In this investigation the action across each of the six faces of the element is normal to that face. In many problems in elastic solids this simplicity does not exist and there are tangential actions also across the faces. For the discussion of these questions the reader is referred to *A Treatise on the Mathematical Theory of Elasticity*, by A. E. H. Love, 1892.

9. Ex. 1. Show that E and $\frac{1}{2}E'$ are the forces which would stretch a rod of unit section to twice its original length and half its original breadth respectively. Show also that E' is greater than $2E$.

If the stretching tension be N_x, v the volume, δv the increase of volume, prove that $$\frac{\delta v}{v}=\frac{E'-2E}{EE'}N_x.$$

Ex. 2. If the side faces of the rod are exposed to a uniform normal pressure equal to p per unit of area, prove that the force required to produce a given extension is less than that in a vacuum by $\lambda p/(\lambda+\mu)$ per unit of area of cross section.

Ex. 3. If a wire be constructed by drawing out a portion of metal, the material is not necessarily isotropic, but we may consider the molecular structure to be symmetrical about any line parallel to the axis of the wire. Assuming that the normal pressures on the faces of the element are given by

$$N_x = a\alpha + f\beta + f\gamma, \quad N_y = f\alpha + b\beta + e\gamma, \quad N_z = f\alpha + e\beta + b\gamma,$$

where $\alpha = du/dx$, $\beta = dv/dy$, $\gamma = dw/dz$ and a, b, e, f are given constants depending on the material of the wire, show that

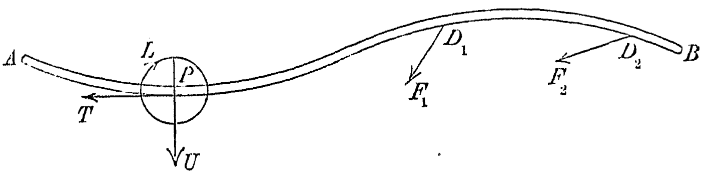
$$\frac{D}{L} = \frac{b+e}{a(b+e) - 2f^2} N_x, \quad \frac{C}{D} = \frac{f}{b+e},$$

where L is the length, D the longitudinal extension and C the lateral compression.

The Bending of Rods.

10. *To form the equations of equilibrium of a thin inextensible rod bent in one plane.*

First Method. In this method we consider the conditions of equilibrium of a finite portion of the rod or wire. The method has been used in Vol. I. Arts. 142—147 to determine the stress at any point of a rod naturally straight and slightly bent by the action of given forces, and the same reasoning may be applied to rods whose natural forms are curved.



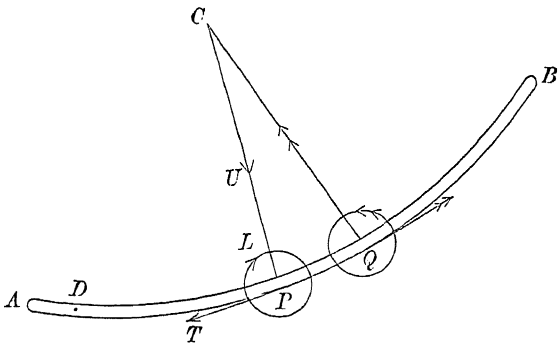
Let P be any section of a thin rod APB regarded as a curved line. Let T and U be the resolved parts of the stress force along the tangent and normal at P, and let L be the stress couple. These represent the mutual action of the two parts AP, PB of the rod on each other. These stresses are then obtained by considering the conditions of equilibrium of the portions AP, PB separately. Let F_1, F_2 &c. be forces acting at the points D_1, D_2 &c. of the portion PB in directions making angles δ_1, δ_2 &c. with the tangent at P. Taking any directions along the tangent and normal at P as positive, let T and U act on the portion PB in these directions; we then have by resolution

$$T + \Sigma F \cos\delta = 0, \quad U + \Sigma F \sin\delta = 0.$$

In the same way if p_1, p_2 &c. be the perpendiculars from P on the lines of action of the forces, we have by moments $L + \Sigma Fp = 0$. These three equations determine T, U and L when the form of the curve is known.

11. *Second Method.* In this method we form the equations of equilibrium of an elementary portion of the rod or wire.

Let PQ be any element of the rod and let the arc s be measured from some fixed point D on the rod up to P in the direction AB so that $s = DP$. Let the stress forces of AP on PB be represented by a tension T acting, when positive, in the direction PA and a shear U acting in the direction opposite to the radius of curvature PC. Then the stress forces of QB on QA are represented by $T + dT$ in the direction QB and $U + dU$ in the direction QC; these directions being represented in the figure by the double arrow heads. Let the stress couple at P on PB be represented by L, the



positive direction being indicated by the arrow head on the circle at P; then the stress couple at Q on AQ is represented by $L + dL$ acting in the opposite direction, i.e. in that indicated by the double arrow heads on the circle at Q. Let Fds, Gds be the impressed forces on the element PQ resolved in the direction of the tangent PQ and normal PC, taken positively when acting respectively in the directions in which the arc s and the radius of curvature ρ are measured. Let $d\psi$ be the angle between the tangents at P and Q, and let ψ be so measured that ψ and s increase together.

Resolving the forces in the direction of the tangent and normal at P, we have

$$-T + (T + dT)\cos d\psi - (U + dU)\sin d\psi + Fds = 0,$$

$$-U + (U + dU)\cos d\psi + (T + dT)\sin d\psi + Gds = 0.$$

In the limit these become

$$dT - Ud\psi + Fds = 0 \quad\text{.........................(1)},$$

$$dU + Td\psi + Gds = 0 \quad\text{.........................(2)}.$$

Also taking moments about P

$$-L + (L + dL) + (U + dU)\,ds + \tfrac{1}{2}\,Gds\,(\tfrac{1}{2}\,ds) = 0,$$

$$\therefore\ dL + Uds = 0 \text{......(3)}.$$

Writing $d\psi = ds/\rho$, these equations take the form

$$\left.\begin{aligned} \frac{dT}{ds} - \frac{U}{\rho} + F &= 0 \\ \frac{T}{\rho} + \frac{dU}{ds} + G &= 0 \\ \frac{dL}{ds} + U \qquad &= 0 \end{aligned}\right\} \text{......................(4)}.$$

If each element of the rod is acted on by an impressed couple, as well as by the impressed forces Fds, Gds, it must be taken account of in the equation of moments. Let Ids be its moment taken positively when the couple acts on the element PQ in the opposite direction to the couple L. We then add Ids to Uds in equation (3) and therefore add I to the left-hand side of the last of equations (4).

12. When we compare the advantages of the two methods of solution we notice that the second gives differential equations which must be integrated, and the constants must be determined by the conditions at the extremities. On the other hand the first method, though it gives expressions for T, U, and L, introduces into the equations the action of all the forces on the finite arc PB. When, therefore, the form of the strained curve is so well known that we can calculate the resolved parts and the moments of the impressed forces the first method gives the required stresses at once. When however the form of the strained curve is very different from that when unstrained, and is itself unknown, the second method presents several advantages over the first.

13. When a thin rod or wire is bent under the action of forces we have to determine not merely the components of stress, i.e. T, U and L, but also the form of the strained rod. The equations of equilibrium found above supply three equations, so that a fourth is required to make up the necessary number. For this purpose we have recourse to experiment, Vol. I., Art. 148. If ρ_1, ρ are the radii of curvature at any point P before and after the deformation, the stress couple L is given by

$$L = K\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) \text{......................(5)},$$

where K is some constant depending on the material of which the rod is made and on the section at P. It is usually called the *flexure rigidity* of the rod.

Since the moment L represents the product of a force and a length, it is evident that the dimensions of K are represented by a force multiplied by the square of a length. If E be Young's modulus for the material of the rod and ω the area of the section, $E\omega$ will represent a force, so that the constant K is often written in the form $K = E\omega k^2$, where k is a length.

It will be shown further on that in certain cases ωk^2 is the moment of inertia of the area of the normal section about a straight line drawn through its centre of gravity perpendicular to the plane of bending.

14. It is hardly necessary to remind the reader of the remarks made in Vol. I. Art. 490, on *the limits to the laws of elasticity*. When the stretching or bending of the rod exceeds a certain limit, the rod does not tend to return to its original form, but assumes a new natural state different from that which it had at first. In all the reasoning in which the equation (5) is used, it is assumed that the bending is not so great that the limit of elasticity has been passed.

15. The theoretical considerations which tend to prove the truth of the equation (5) depend on the theory of elasticity and therefore lie somewhat outside the scope of the present chapter. As however this theory clears up some of the difficulties which belong to the bending of rods, it does not seem proper wholly to pass it over. One case can be presented in a simple form, and that case will be discussed a little further on after the use of the equation (5) has been explained.

16. **The work of bending an element.** *To find the work done by the stress couple L when the curvature of an element of the rod is increased from its natural value $1/\rho_1$ to the value $1/\rho_2$.*

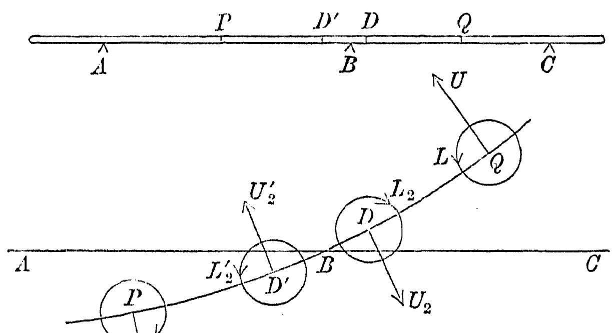
Let PQ be an element of the central line and let ds be its length. As PQ is being bent, let ψ be the angle between the tangents at its extremities; let ρ be its radius of curvature. If ψ_1 be the value of ψ when the rod has its natural form, the stress couple L is
$$L = K\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) = K\frac{\psi - \psi_1}{ds}.$$
The work done by the couple L when ψ is increased by $d\psi$ is $-Ld\psi$, (see Vol. I. Art. 291). The negative sign is given to the expression because, as explained in Art. 11, L is measured in the direction opposite to that in which ψ is measured. The whole work done by the couple when ψ is increased from ψ_1 to ψ_2 is therefore equal to
$$-\tfrac{1}{2}K \cdot \frac{(\psi_2 - \psi_1)^2}{ds}.$$

Replacing ψ_2, ψ_1 by their values in terms of ρ_2, ρ_1, we see that the work Wds done by the couple L may be written in either of the forms
$$Wds = -\tfrac{1}{2}K\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)^2 ds = -\frac{L^2 ds}{2K}.$$

If the change of curvature at every point of the rod is known, the whole work done by the stress couples in the rod may be found by integrating the first of these expressions along the length of the rod. If however the change of curvature is unknown, and the couple is given, the work is found by integrating the latter expression.

17. **Deflection of a straight rod.** *A heavy rod rests on several supports arranged in a horizontal straight line, and is slightly deflected by its own weight. It is required to explain the method of finding the deflection at any point of the rod and to determine the relations which exist between the stresses at successive points of support.*

Let A, B, C be three successive points of support, let $AB = a$, $BC = b$. Let x be measured from B in the direction BC. The rod, when bent by its weight, will assume the form of some curve which differs very slightly from the straight line ABC. Let y be the ordinate at any point Q between B and C measured positively upwards, and let the radius of curvature be positive when the concavity is upwards. The stress couple at the point Q is K/ρ; when ρ is positive the fibres of the under part of the rod are stretched while those above are compressed, hence the stress couple at Q acts on QC in the clock direction and on BQ in the opposite direction. Let the shear at Q be U and let its positive direction when acting on QC be downwards.



Let L_2 and U_2 be the couple and shear at a point D indefinitely near to B on its right-hand side. Let w be the weight of the rod per unit of length, then the weight of DQ is wx, and this weight acts at the centre of gravity of DQ. Taking moments about Q for the finite portion of the rod DQ, we have

$$\frac{K}{\rho} = L_2 - U_2 x - \tfrac{1}{2}wx^2 \text{ ....(1).}$$

In forming the right-hand side of this equation the rod has been supposed to be straight, because the deflections are so small that only a very small error is made by neglecting the curvature. If this were not so, the shear would not be vertical, and the arm of its moment would be different from that used in the equation. In the same way the thickness of the rod has been neglected, and in all its geometrical relations the rod is regarded as if it were a line coincident with its central axis, Art. 1.

The rod is supposed to be of such material that a considerable effort is required to produce a slight curvature; the coefficient K is therefore large. On the left-hand side of the equation all the small terms cannot be rejected because these are multiplied by K. It is however sufficient, in a first approximation, to retain only the largest of these small terms. We therefore put

$$\frac{1}{\rho} = \pm \frac{d^2y}{dx^2}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{-\frac{3}{2}} = \pm \frac{d^2y}{dx^2}.$$

The upper sign must be taken because ρ is measured positively when the concavity is upwards, and in this case dy/dx is increasing and therefore d^2y/dx^2 is positive.

The general rule followed in these problems is, (1) that all terms not containing K are formed on the supposition that the rod has its natural shape, (2) that in all terms containing K as a factor only the first power of the deflection y is retained.

18. The equation (1) now takes the form

$$K\frac{d^2y}{dx^2} = L_2 - U_2 x - \tfrac{1}{2}wx^2 \text{....................(2),}$$

where x is restricted to lie between $x = 0$ and $x = b$. Let L_2' and U_2' be the stress couple and shear at a point D' indefinitely near B on its left-hand side, and let R_2 be the pressure of the point of support B on the rod inwards. By considering the equilibrium of the small portion $D'D$ of the rod we have by moments and resolution $\qquad L_2' = L_2, \quad U_2' - U_2 = R_2$ ....(3).

If we take a point P between A and B so that BP represents a negative value of x, we have $$K\frac{d^2y}{dx^2} = L_2' - U_2'x - \tfrac{1}{2}wx^2 \ldots\ldots (4),$$ where x is restricted to lie between $x = 0$ and $x = -a$.

Lastly, if U be the shear at any point of the rod we have by equation (3) of Art. 11 $$U = -\frac{dL}{dx} = -K\frac{d^3y}{dx^3} \ldots\ldots\ldots\ldots\ldots (5).$$

It is evident that the two arcs AB, BC of the rod must have the same tangent at B and therefore the same value of dy/dx. It follows from the first of equations (3) that the stress couples on each side of B are equal; the two arcs have therefore the same curvature. But the shears on each side of B differ by the pressure R_2, and therefore there is an abrupt change in the value of d^3y/dx^3 at a point of support.

Integrating (2) we have, if β be the inclination of the rod at B to the horizon, $$K\frac{dy}{dx} = K\beta + L_2x - \tfrac{1}{2}U_2x^2 - \tfrac{1}{6}wx^3 \ldots\ldots\ldots (6),$$
$$Ky = K\beta x + \tfrac{1}{2}L_2x^2 - \tfrac{1}{6}U_2x^3 - \tfrac{1}{24}wx^4 \ldots\ldots\ldots (7),$$
the constant in the last equation being omitted because x and y vanish together.

Since $y = 0$ when $x = b$, we also have
$$0 = K\beta + \tfrac{1}{2}L_2b - \tfrac{1}{6}U_2b^2 - \tfrac{1}{24}wb^3 \ldots\ldots\ldots\ldots\ldots (8).$$
Integrating (4) in the same way, we have
$$0 = K\beta - \tfrac{1}{2}L_2'a - \tfrac{1}{6}U_2'a^2 + \tfrac{1}{24}wa^3 \ldots\ldots\ldots\ldots\ldots (9).$$

19. **Equation of the three moments.** If L_1, L_2, L_3 be the stress couples at the three successive points of support A, B, C we have by taking moments about C and A
$$L_3 = L_2 - U_2b - \tfrac{1}{2}wb^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$
$$L_1 = L_2 + U_2'a - \tfrac{1}{2}wa^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$
Eliminating U_2, U_2', and β from (8), (9), (10) and (11), we have
$$L_1a + 2L_2(a+b) + L_3b + \tfrac{1}{4}w(a^3+b^3) = 0 \ldots\ldots\ldots (12).$$

This important relation between the stress couples at any three successive points of support is usually called *the equation of the three moments*. By help of this theorem, when the stress couples at two of the points of support are known, the stress couples at all the points may be found. The shears are then determined by (10) and (11) and the pressures on the points of support by (3).

20. *Extension of the theorem of three moments to the case in*

which the three points of support are only nearly on a level. If y_1, y_3 be the altitudes of the points of support A and C above B, we may prove in the same way

$$6K\left(\frac{y_3}{b}+\frac{y_1}{a}\right)=L_1a+2L_2(a+b)+L_3b+\frac{w}{4}(a^3+b^3)\ldots(13).$$

It is assumed that the differences of level of the three points of support are of the same order of small quantities as the deflection of the rod.

In this form the equation of the three moments may be regarded as the relation between the ordinates y_1 and y_3 at any two points, and the stress couples at these points. The equation therefore gives the ordinate y_3 at any point at which the stress couple L_3 is known; for example at the free end, where $L_3=0$.

21. If the rod rest on n points of support, the equation of the three moments supplies $n-2$ equations connecting the n stress couples $L_1, L_2, \ldots L_n$ at the points of support. Two more equations are therefore necessary to find the n couples, and these may be deduced from the conditions at the extremities.

If one end of the rod is free, and at a distance c from the nearest point of support, the stress couple L_n at that point of support is found, by taking moments about it, to be $L_n=-\frac{1}{2}wc^2$.

If an extremity rest on a point of support the stress couple at that point is zero.

If an extremity be built into a wall so that the tangent to the rod at that point is fixed in a horizontal position we may imagine that the fixture is effected by attaching that end of the rod to two points of support indefinitely close together. The required condition at that end then follows at once from the equation of the three moments. Let L_{n+1} be the couple at the wall, L_n that at the nearest point of support and let c be the distance, then writing $a=c$, $b=0$ in the equation of the three moments we have

$$L_n+2L_{n+1}+\tfrac{1}{4}wc^2=0.$$

The pressures on the points of support may be obtained by combining equations (10), (11) with (3). If R_2 be the pressure on the rod measured *upwards* at B, we find by eliminating U_2, U_2'

$$\frac{L_1}{a}-L_2\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{L_3}{b}=R_2-\tfrac{1}{2}w(a+b)\ldots\ldots\ldots(14).$$

This result has also been attained in Vol. I. Art. 145.

If weights are fastened at any given points of the rod, these may be regarded as points of support at which the pressure is known. The deflection at each of these points being unknown, the extended equation of the three moments fails to determine the stress couple. But the pressure being known, the equation (14) gives an additional equation connecting the stress couples, and the extended equation of the three moments then gives the deflection.

22. Ex. 1. A uniform heavy rod is supported at its extremities; the deflection at its middle point is observed and found to be h. Show that the value of the constant K for the rod is given by $K=\dfrac{5a^3W}{48h}$, where $2a$ is the length of the rod. If the inclination to the horizon of the tangent at either end of the rod be observed by a level and found to be θ, show that the value of K is also given by $K=\dfrac{a^2W}{6\theta}$.

This example shows how the value of K may be found by experiment for any given rod.

Ex. 2. A uniform heavy rod is supported at its extremities A, C and at its middle point B; A and C are at the same level and B such that the pressures on the three supports are equal. Prove that the depth of B below AC is 7/15ths of the whole central deflection of the beam AC when supported only at its ends.

This example shows that when a long heavy bridge is supported on three columns of equal strength, their summits ought not to be on the same level.

Ex. 3. A heavy rod rests on a series of points of support which are very nearly in a horizontal line. Let A, B be any consecutive two of these points, a their distance apart, y_1, y_2 their altitudes above a horizontal plane. Let L_1, L_2 be the stress couples, θ_1, θ_2 the inclinations of the rod to the horizon at A, B. Prove that

$$K(\tan\theta_2-\tan\theta_1)=\tfrac{1}{2}(L_1+L_2)\,a+\tfrac{1}{12}wa^3,$$
$$K(y_2-y_1-a\tan\theta_1)=\tfrac{1}{6}(2L_1+L_2)\,a^2+\tfrac{1}{24}wa^4.$$

The stress couples having been found, the first of these equations enables us to find the inclination of the rod at any point of discontinuity when the inclination at some point is known. The second determines the inclination at any one point.

23. Ex. 1. A uniform heavy beam ABC is supported at its extremities A, C and at its middle point B, and the three points are in one horizontal line. Prove that 3/16ths of the weight is supported at either end and 5/8ths at the middle point. We notice that the pressure at the middle support is more than three times that at either end.

Prove also that the stress couple is a maximum at a point which divides either span in the ratio of 3:5, but the stress couple at either of these points is 9/16ths of the stress couple at the central point of support. Prove that the latter is equal to the stress couple at the middle point of a beam supported at each end whose length is equal to that of either span.

Prove that there is a point of contrary flexure in each span dividing it in the ratio 1 : 3.

Ex. 2. A uniform beam is supported at its extremities and at two other points dividing the beam into three equal spans, all the four points being on the same level. Prove that the pressures on the supports are in the ratios 4 : 11 : 11 : 4.

Ex. 3. A uniform beam $ABCDE$ is supported at its extremities A, E and at three points B, C, D, all five being on the same horizontal line. To assimilate this problem in some measure to the case of the Britannia Bridge the two middle spans are supposed to be twice the lengths of the outside ones, i.e. $BC=CD=2AB=2DE$. Prove that the pressures on A, B, C are in the ratios 4 : 27 : 34.

The examples in this article are taken from a treatise on *The Britannia and Conway tubular bridges* by Edwin Clark, resident engineer, 1850.

The tubes AB, BC, CD, DE, which form the four spans of the Britannia Bridge, were raised separately into their proper places and then rigidly connected into one long tube. The connections at B and D were such that the tubes adjacent to each had a common tangent. The junction at C was however so arranged that the tangents to BC and CD should make a small angle with each other. The object of this was to diminish the inequality between the pressure on C and that on either B or D. It was found convenient to make the angle between the tangents equal to $2\tan^{-1}\cdot002$. In Example 3, given above, the analytical condition to be satisfied at C is that the tangents to AC and CB should be continuous, but in the bridge the condition is that these tangents should make a known small angle with each other.

24. Ex. 1. A rod without weight is supported at its extremities A, C and at some other point B, all three being in the same horizontal line. Given weights P, Q are suspended at the points D, E, bisecting AB and BC. Show that the inclination to the horizon of the tangent at A and the deflection y at the weight P are given by

$$32(a+b)K\tan\alpha=-Pa^2(a+2b)+Qab^2,$$
$$768(a+b)Ky=-P(7a+16b)a^3+9Qa^2b^2,$$

where $AB=a$, $BC=b$.

It appears from this result that when the point of support B bisects AC and $Q=3P$ the tangent at A should be horizontal. Moseley describes three experiments with different rods supported on knife edges by which this curious result has been verified. See his *Mechanical Principles of Engineering and Architecture*, 1855, page 527.

Ex. 2. A uniform thin rod of length $2(a+b)$ rests on two points of support in a horizontal line whose distance apart is $2a$. Show that, if the middle point and the two free ends are on the same horizontal line, b/a must be the positive root of the cubic $3r^3+9r^2-3r-5=0$.

25. Ex. 1. A uniform heavy rod rests on any number of points of support in the same horizontal line. Let A, B, C, D, E be any consecutive five of these, and let their distances apart be a, b, c, d. Prove that the pressures R_2, R_3, R_4, at B, C, D are connected by the linear relation $\alpha R_2+\beta R_3+\gamma R_4=\frac{1}{4}w\delta$, where

$$\alpha=a^2(b+c)(c+d)(b+c+d),$$
$$\beta=(a+b)(c+d)\{b^2(d+2c)+2bc(a+d)+c^2(a+2b)+ad(b+c)\},$$
$$\gamma=d^2(b+c)(a+b)(a+b+c),$$
$$\delta=(a+b)(b+c)(c+d)(b+c+d)(a+b+c)(a+b+c+d).$$

Ex. 2. Prove that the deflection y at any point Q between B and C is given by

$$-6Kby=BQ\cdot CQ\{L_2(b+CQ)+L_3(b+BQ)+\tfrac{1}{4}wb(b^2+BQ\cdot CQ)\},$$

where $BC=b$.

26. Ex. 1. A wire is bent into the form of a circle of radius c, and the tendency at every point to become straight varies as the curvature. Show that, if it be made to rotate about any diameter with a small angular velocity ω, it will assume the form

of an ellipse whose axes are $2c\left(1 \pm \frac{m\omega^2 c^4}{12\mu}\right)$, m being the mass of a unit of length, and μ/c the couple necessary to bend the straight line into the circle.

[Math. Tripos, 1868.]

Ex. 2. A heavy elastic flexible wire originally straight is soldered perpendicularly into a vertical wall. If the deflection is not small prove that the difference between the tension at any point P and the weight of a portion of the wire whose length is the height of P above the free end is proportional to the square of the curvature at P. [May Exam.]

Ex. 3. A flexible wire is pushed into a smooth tube forming an arc of a circle, and lies in a principal plane of the tube; prove that it will only touch it in a series of isolated points, and that if it only touch the inner circumference at one point, the pressure there will be $4E \cos \alpha (\sin \alpha - \sin \gamma)/a^2 \sin^2 \alpha$, where a is the inner radius of the tube, 2α the angle subtended at the centre by the wire, γ the angle at which either end of it meets the wire, and E the coefficient of flexibility.

[Math. Tripos, 1871.]

Ex. 4. Three very slightly flexible rods are hinged at the extremities so as to form a triangle, and are attracted by a centre of force attracting according to the law of nature situated in the centre of the inscribed circle. Shew that the curvature of any side as AB at the point of contact of the inscribed circle varies as

$$\frac{\cos \frac{1}{2}A + \cos \frac{1}{2}B - \cos \frac{1}{2}C}{\cos \frac{1}{2}C}.$$

Ex. 5. Equal distances AB, BC, CD are measured along a light rod which is supported horizontally by pegs at B, D below the rod and C above. A weight is now hung on at A, producing at that point a deflection. Find how much B must be moved horizontally towards A that the deflection may be unaltered when the peg D is removed. [Coll. Exam. 1888.]

27. Ex. 1. A uniform heavy rod rests symmetrically on $2m+1$ supports placed at equal distances apart, and the altitudes are such that the weight of the rod is equally distributed over the supports. Show that the altitude y_p of the support distant pa from the middle point is given by

$$\frac{24(2m+1)K}{wa^4}(y_p - y_0) = (2\beta - 1)p^4 - \{(2\beta-1)(6m^2-1) + (6\beta^2-1)(2m+1)\}p^2,$$

where a is the distance between two consecutive points of support and βa is the length of the rod beyond either of the terminal supports.

We first see by taking moments about the pth support that the stress couple L_p at that point is a quadratic function of p. The extended equation of the three moments is $\quad L_p + 4L_{p+1} + L_{p+2} + \frac{1}{2}wa^2 = (y_{p+2} - 2y_{p+1} + y_p)\, 6K/a^2$.

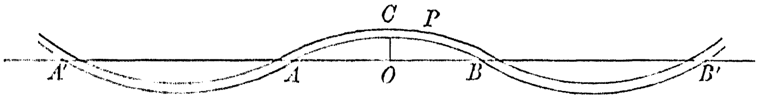
By an easy finite integration, or by the rules of algebra, it follows that y_p is a biquadratic function of p. Since there can be no odd powers of p, we have

$$y_p - y_0 = Ap^4 + Bp^2.$$

The values of A and B are then found by applying the equation of the three moments to any two convenient spans.

28. **A bent bow.** *A uniform inextensible rod, used as a bow, is slightly bent by a string tied to its extremities. It is required to find its form.*

Taking the string as the axis of x, the statical equation is evidently
$$\frac{K}{\rho} = \pm K\frac{d^2y}{dx^2} = Ty \quad\text{.................... (1)},$$
where T is the tension of the string. Let A, B be the extremities of the rod, C a point on the rod at which the tangent is parallel to the string. Let OC be the axis of y. Then since dy/dx vanishes when $x=0$ and decreases algebraically as x increases, d^2y/dx^2 is negative. In forming (1) ρ has been taken as positive, we must therefore give the second term the negative sign. Putting $T=Kn^2$ for brevity, the equation gives
$$y = h\cos nx \quad\text{.............. (2)},$$



where h is the versine of the arc formed by the bow. It is obvious that unless the conditions of the problem make h small, we cannot reject the terms containing $(dy/dx)^2$ in the expression for ρ in equation (1).

The form of the curve given by the equation (2) is sketched in the diagram. It appears therefore that the bow may take the form ACB, the string being attached at A and B. It may also take the form ACB' with the string attached at A and B', and so on.

Let the length of the bow be $2l$ and that of the string $2a$, then a and l are nearly equal. We have $y=0$ when $x=a$, hence
$$na = \tfrac{1}{2}\pi(2i+1), \qquad T = \frac{\pi^2 K}{4a^2}(2i+1)^2 \quad\text{........... (3)},$$
where i is an integer whose value depends on which of the points A, A', &c. B, B', &c. are taken as the terminals of the bow. We also have
$$l = \int\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx = a\left(1+\tfrac{1}{4}h^2n^2\right) \quad\text{........(4)},$$
nearly, the limits of integration being 0 and a. If ACB be the rod, this gives $\pi^2h^2 = 16a(l-a)$.

29. By considering only the half CB of the bow the same analysis gives *the form of a uniform rod CB having one end C and the tangent at C fixed while the other end B is acted on by a force whose direction is parallel to the tangent at C.*

Regarding the force T as given, n is known; since y must be zero when $x=a$, we must have either $h=0$ or $na=\frac{1}{2}\pi(2i+1)$, where i is any integer.

Supposing the length l of the rod to be given, it follows that it is only when the force is nearly equal to $\dfrac{\pi^2 K}{4l^2}(2i+1)^2$ that there can be a small deflection. If the force have other values, either $h=0$ and there is no deflection, or the deflection is so great that the terms containing dy/dx in equation (1) cannot be rejected.

30. When the terms containing dy/dx are included in equation (1), we have
$$-K\frac{y''}{(1+y'^2)^{\frac{3}{2}}}=Ty \quad\text{.................... (5)},$$
where accents denote differential coefficients with regard to x. Multiplying by y' and integrating, we find
$$1-\cos\psi=\tfrac{1}{2}n^2(h^2-y^2) \quad\text{.................. (6)},$$
where ψ is the acute angle made by the tangent at any point P with the string of the bow.

Referring to the diagram, let β be the acute angle made by the tangent at either A or B with the string. Then since (6) is satisfied by $y=0$, $\psi=\beta$, we find $\quad 2\sin\tfrac{1}{2}\beta=nh$ .... (7).

The equation (5) may be written in the form $\dfrac{d\psi}{ds}=n^2y$. Substituting for y and integrating between the limits $s=0$ and $s=l$, we have
$$l=\frac{1}{2n}\int_0^\beta\frac{d\psi}{(\sin^2\frac{1}{2}\beta-\sin^2\frac{1}{2}\psi)^{\frac{1}{2}}} \quad\text{.............. (8)}.$$

We may notice that if the integration extend beyond the point B in the diagram, both numerator and denominator simultaneously change sign at B. If the length of the half bow CB, and the tension or force at B, are given, both l and n are known. The equations (7) and (8) may then be used to find β and the deflection h.

The integral is lessened by introducing the factor $\cos\frac{1}{2}\psi$ into the numerator. We therefore have
$$nl>\tfrac{1}{2}\int\frac{\cos\frac{1}{2}\psi\,d\psi}{(\sin^2\frac{1}{2}\beta-\sin^2\frac{1}{2}\psi)^{\frac{1}{2}}}\quad\text{i.e.}>\left[\sin^{-1}\frac{\sin\frac{1}{2}\psi}{\sin\frac{1}{2}\beta}\right]_0^\beta.$$
It immediately follows that $nl>\frac{1}{2}\pi$. Thus unless the tension or force exceed the value $\pi^2K/4l^2$ the equation (8) cannot be satisfied. It follows that ψ cannot be taken as the independent variable, i.e. there is no deflection.

31. The importance of the case considered in Art. 29 lies in its application to the theory of thin vertical columns. The rod may be regarded as a vertical column having the tangent at its

lower end C fixed in a vertical position, while a weight, much greater than that of the column, is supported on the upper extremity. It appears from what precedes that if the weight on the summit is gradually increased, the column will remain erect, without bending, until the weight becomes nearly equal to a certain quantity depending on the flexibility and dimensions of the column.

Since the constant K is equal to $E\omega k^2$ (Art. 13) it follows that the bending weight, for columns of the same kind, varies as the fourth power of the diameter directly, and as the square of the length inversely. This result is usually called Euler's* law.

Columns yield under pressure in two ways, first the materials may be crushed, and secondly the column may bend and then break across. In some cases both effects may occur at once. If the column is short it follows from Euler's law that the bending weight is large, so that short columns yield by crushing. Long columns on the other hand break by bending and are not crushed.

Many experiments have been made to test the truth of Euler's law. The results have not been altogether confirmatory, possibly because Euler's law applies only to uniform thin columns, in which the central line in the unstrained state is a vertical straight line. For the details of these experiments we must refer the reader to works on engineering. See also Mr Hodgkinson's *Experimental researches on the strength of pillars*, Phil. Trans. 1840.

32. Ex. 1. A vertical column in the form of a paraboloid of latus rectum $4m$ with its vertex upwards is fixed in the ground. Show that it will bend under its own weight when slightly displaced if the length be greater than $\pi(2Em/w)^{\frac{1}{2}}$, where w is the weight of a unit of volume, E the weight which would stretch a bar of the same material and unit area to twice its natural length.

Suppose the column slightly bent, consider any horizontal section say at the point P of the central line. If ρ be the radius of curvature, ωk^2 the moment of inertia of the section, the equation of equilibrium is obtained by equating $E\omega k^2/\rho$ to the moment of the weight of that part of the column which lies above P.

This example and the next are taken from a paper by Prof. Greenhill.

Ex. 2. A vertical cylindrical column of radius a is fixed in the ground. Show that it will bend under its own weight if its length h be greater than $c^{\frac{2}{3}}\left(\frac{9Ea^2}{16w}\right)^{\frac{1}{3}}$, where c is the least root of $J_{-\frac{1}{3}}(c)=0$, w the weight of a unit of volume and E the weight which will stretch a column of unit section to twice its length.

33. **Theory of a bent rod.** *A uniform thin straight rod is bent without tension*

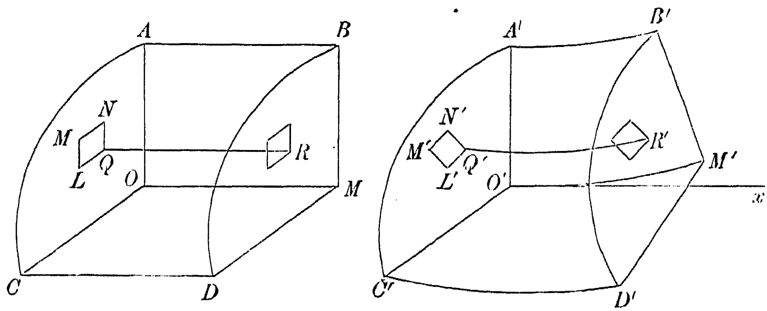
* Euler, *Berlin Memoirs*, 1757. *Petersburg Commentaries*, 1778. Lagrange, *Acad. de Berlin*, 1769. Poisson, *Traité de Mécanique*, 1833.

into the form of a circular arc of great radius; it is required to find the stress couple at any point P.

We shall obtain a particular solution of this problem by making an hypothesis which simplifies the process, and which we afterwards verify by showing that all the equations of equilibrium are satisfied.

We assume (1) that all filaments of matter parallel to the length of the rod are bent into circles with their centres on a straight line perpendicular to the plane of bending. This straight line will be referred to as the *axis of bending*. We assume (2) that the particles of matter which in the unstrained rod lie in a normal section continue to lie in a plane when bent, (3) that this plane is normal to the system of circles above described.

Let $ABCD$ be a short length of the straight rod bounded by two normal planes AOC, BMD. To examine the small changes which this length undergoes we take the plane AOC as that of yz and let some perpendicular straight line OM be the axis of x. To avoid confusing the figure only the lines on the positive octant have been drawn. Let the plane of xz be the plane of bending, so that the axis of y is parallel to the axis of bending. Thus OA is the axis of z, OC that of y. Let QR be any elementary filament parallel to the axis of x, let $(0, y, z)$, (x, y, z) be the coordinates



of Q and R. Let the positions of these points and lines in the bent rod be denoted by corresponding letters with accents. According to the hypothesis $A'O'C'$, $B'M'D'$ are normal to all the filaments of the bent rod, and (when produced) these planes intersect in the axis of bending. Any filament, such as $Q'R'$, is a circular arc whose unstretched length is OM.

The rod being bent without tension, the filaments near $A'B'$ are compressed while those on the opposite side of the rod are extended. There is therefore some surface such that the filaments which lie on it have their natural length. This surface is usually called *the neutral surface*, and the lines on it parallel to the length of the rod are called *neutral lines*. Since the filaments on this surface are circular arcs of the same length with their centres on the axis of bending, the neutral surface is a cylinder which cuts the plane of yz in a straight line parallel to the axis of bending. Let the origin O' be taken on the neutral surface, the axis of x is therefore a tangent to a neutral line, and the unstretched length of every filament, such as $Q'R'$, is equal to OM or OM'. Let ρ be the radius of curvature of this neutral line. Since the rod is thin, all the linear dimensions of the mass $ABCD$ are small compared with ρ.

When the unstretched length QR has been compressed or stretched into the length $Q'R'$, it remains sensibly parallel to the axis of x, but its distances from the planes xz, xy may have been altered. Let these distances be $y' = y + v$, $z' = z + w$, and

let the stretched length $Q'R'$ be $x'=x+u$. Since R' lies in a plane normal to the neutral line at M', we have $\quad x'=(\rho-z-w)\sin\dfrac{x}{\rho}=x-\dfrac{(z+w)\,x}{\rho}$.

The difference $x'-x$ represents the stretch of the fibre QR whose unstretched length is x. The tension per unit of sectional area is therefore equal to $-E\dfrac{z+w}{\rho}$. The displacement w being small compared with z, we may, as a first approximation, equate the tension to $-Ez/\rho$.

Since the resultant tension across the section AOC is zero, we have

$$\iint (Ez/\rho)\,dy\,dz=0.$$

It immediately follows that the centre of gravity of the section lies in the plane of xy. The neutral surface therefore passes through the centre of gravity of every normal section. *In a cylindrical rod therefore, bent without tension, the central line is also a neutral line.*

Since the elementary tensions have no components parallel to the axes of y or z, it follows that the shear is zero.

If L be the moment about the axis of y of the tensions which act across the section AOC, measured positively from z to x, we have

$$L=\iint z^2\,dy\,dz\,.\,\frac{E}{\rho}=E\,\frac{\omega k^2}{\rho},$$

where ωk^2 is the moment of inertia of the sectional area about the axis of y, i.e. about a straight line drawn through the centre of gravity of the section perpendicular to the plane of bending, see Art. 13. Since the rod is a uniform cylinder bent into a circular arc, the corresponding couples about $O'C'$, $M'D'$ balance each other.

In the same way the moment about the axis of z of the tensions which act across the section AOC is $\iint yz\,dy\,dz\,.\,E/\rho$. This couple cannot be balanced by the equal couple about $M'B'$ because their axes are not parallel. It is therefore necessary that this moment should vanish. It follows that the rod will not remain in the plane of bending unless the product of inertia of the area of the normal section about the axis of y and any perpendicular straight line in its plane is zero. In other words, the plane of bending must be perpendicular to a principal axis of the section at its centre of gravity.

34. If we suppose, as already explained in Art. 8, that each fibre or filament of the rod is contracted or extended in the same manner as if it were separated from the rest of the rod, the mutual pressures of these filaments transverse to the length of the rod and also the tangential actions are zero. Each element of the rod is therefore in equilibrium, and the surface conditions are also satisfied. Each filament is slightly displaced, like those discussed in Art. 8, and slightly turned round. These displacements are those represented by v, w, and are such that, when the fibres are stretched independently of each other, the body remains continuous.

The expressions for the coordinates $y'=y+v$, $z'=z+w$, of Q' in terms of the coordinates y, z of Q may be deduced from the theorems given in Art. 8. It follows from that article that when the filament QR is stretched into the filament $Q'R'$ by a tension N_x, the rectangular base $QLMN$ remains rectangular and similar to its original form, and is of such size that corresponding sides are connected by the relation $(Q'L'-QL)/QL=-N_x/E'$.

Let ϕ be the angle which the side $Q'L'$ makes with the axis of y measured positively from z to y; then

$$Q'L'\cos\phi=\frac{dy'}{dy}\,dy=\left(1+\frac{dv}{dy}\right)dy,$$

$$-Q'L'\sin\phi=\frac{dz'}{dy}dy=\frac{dw}{dy}dy.$$

Rejecting the squares of the small quantities v, w and remembering that $QL=dy$, these give
$$\frac{dv}{dy}=-\frac{N_x}{E'},\qquad -\tan\phi=\frac{dw}{dy}.$$

Treating the side $Q'N'$ in the same way, we have
$$\frac{dw}{dz}=-\frac{N_x}{E'},\qquad \tan\phi=\frac{dv}{dz}.$$

Substituting for N_x its value $-(z+w)/\rho$, and neglecting w/ρ as before, we find by integration
$$v=\frac{E}{E'}\frac{yz+f(z)}{\rho},\qquad w=\frac{E}{2E'}\frac{z^2+F(y)}{\rho}.$$

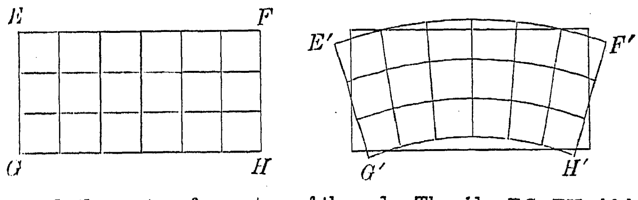
Equating the two values of ϕ and substituting for v and w, we find that
$$-\tfrac{1}{2}F'(y)=y+f'(z).$$

It follows that $f(z)=az+b$, and therefore
$$v=\frac{E}{E'}\frac{(y+a)z+b}{\rho},\qquad w=\frac{E}{2E'}\frac{z^2-(y+a)^2-c}{\rho}.$$

The terms independent of y, z represent a translation of the section as a whole, those containing the first powers represent a rotation through an angle $Ea/E'\rho$. If neither of these displacements exist, we may omit these terms.

The expressions thus found for u, v, w, give the displacements of Q referred to the axes $O'M'$, $O'A'$, $O'C'$. They also give those of R referred to corresponding axes with M' for origin. The displacements of R referred to the axes with O' as origin are therefore given by
$$u=-\frac{xz}{\rho},\qquad v=\frac{E}{E'}\frac{yz}{\rho},\qquad w=\frac{x^2}{2\rho}+\frac{E}{E'}\frac{z^2-y^2}{2\rho},$$
where x, y, z are the coordinates of R.

35. If the section of the beam is a rectangle having the sides EF, GH perpendicular to the plane of bending, we see by examining the expression for v and w that these sides become curved when the rod is bent, and that they have their convexities



turned towards the centre of curvature of the rod. The sides EG, FH which before bending were parallel to the plane of bending remain straight lines but are inclined to the plane of bending and tend outwards on the concave side of the rod.

The expressions found in Art. 34 for the displacements u, v, w agree with those given by Saint-Venant for one case of bending. But what has been said in that article is not to be taken for a complete discussion of his problem; for that the reader should consult a treatise on the theory of Elasticity.

36. **Airy's Problem.** In using standards of length two considerations have attracted attention, (1) the application of supports in such a manner as to produce no irregularities of flexure and (2) the application of such supports as will permit the expansive or contractive effects of temperature. The importance of the former

was made known by Kater, that of the latter by Baily. Freedom of expansion is usually secured by supporting the body on rollers. Excessive flexure is avoided by making the rollers rest on levers which are so arranged that the weight of the body is either equally distributed over the points of support or distributed in such ratios as may be thought proper.

The flexure is so small that the mere curvature of the central line does not produce a sensible alteration of its length. If however the measured length is on the upper surface of the standard it is assumed that the extension of each element of its length is proportional to the bending moment. If L be the bending moment and dx an element of length, Airy's principle is that the rod should be so supported that $\int L dx$ is zero, the limits of the integration being from one end of the measured length to the other.

We may deduce the correctness of this principle from the theory given in Art. 33. The extension of the filament QR has been shown to be approximately $QR\,(z/\rho)$, where ρ is the radius of curvature of the central line and z the distance from the central line of the projection of QR on the plane of bending. If then z be the half thickness of the rod, the extension of an element dx on the surface is $z dx/\rho$. Since $L = K/\rho$, it immediately follows that the extension of any element on the surface of a uniform rod is proportional to the bending moment.

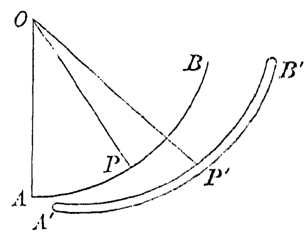
Ex. 1. A bar, of length a, is supported at two points symmetrically placed, and the marks defining the extremities of the measured length are close to its ends; prove that the distance between the points of support should be $a/\sqrt{3}$.

Ex. 2. A standard of length a is supported on n rollers placed at equal distances, and the weight is equally distributed over the rollers. The measuring marks are placed at distances e from the ends. If D be the distance between two consecutive rollers, prove that $$D\sqrt{(n^2-1)} = a\sqrt{(1-8e^3/a^3)}.$$

Memoirs of the Royal Astronomical Society, Vol. xv., 1846, and *Monthly Notices*, Vol. vi., 1845.

37. Bending of Circular rods. *The natural form of a thin inextensible rod is a circular arc; supposing it to be slightly flexible, it is required to find the deviation from the circular form produced by any forces*.*

Let AB be the arc of the circle when undeformed, O its centre, a its radius. Let P be any point on the circle, P' the corresponding point on the rod when bent. Let a, θ be the polar coordinates of P; $a(1+u)$, $\theta+\phi$ those of P', referred to O as origin.



If ρ be the radius of curvature at P', we have by a theorem in the differential calculus

$$\frac{1}{\rho} - \frac{1}{a} = -\frac{1}{a}\left(u + \frac{d^2u}{d\theta^2}\right) \quad\text{..............(1)},$$

* The case of a circular arc is important because the periods of its vibrations, both when inextensible and extensible, can be found. See the second volume of the Author's *Rigid Dynamics*, where also the expression for the work of the stresses is found in a different manner.

where the squares of u are neglected. Let us represent either side of this equation by q/a.

If the central line be extensible, let ds_1 and ds be the unstretched and stretched lengths of an element of arc. We then have

$$ds_1 = ad\theta,$$
$$(ds)^2 = (adu)^2 + a^2(1+u)^2(d\theta + d\phi)^2.$$

Neglecting the squares of small quantities, this gives

$$ds = a(1+u)\,d\theta + ad\phi.$$

If p be the proportional elongation of the elementary arc

$$p = \frac{ds - ds_1}{ds_1} = \frac{d\phi}{d\theta} + u \quad \text{.................(2)}.$$

If the rod is inextensible, we have $p = 0$.

The equations of equilibrium of an inextensible rod may be formed by either of the methods described in Arts. 10, 11. Taking, for example, the three equations marked (4) in Art. 11, and joining them to

$$L = K\frac{q}{a}, \quad p = 0 \quad \text{.................(3)},$$

we have five equations to find T, U, L, u, ϕ in terms of θ.

38. If the rod is slightly extensible as well as flexible, the equations become somewhat changed. The arc ds in the equations of equilibrium in Art. 11 means now the stretched length of the element, while F and G represent the impressed forces referred to a unit of length of the stretched rod. The equation $p=0$ must also be replaced by another connecting p with the tension.

The relations which connect L and T with p and q are perhaps most easily deduced from the expression for the work done by the stresses when the rod is deformed. If Wds_1 be the work done by the stresses when the element is stretched and bent, we have

$$Wds_1 = -\tfrac{1}{2}ds_1\left(Hp^2 + \frac{Kq^2}{a^2}\right) \quad \text{.........................(4)},$$

where H and K are the constants of tension and flexural rigidity. This result follows at once from those given in Art. 16 of this volume and in Art. 493 of Vol. I., when we assume that the work due to a deformation of bending is independent of that of stretching.

From this expression for W we may deduce the values of T and L. Keeping one end P' of an element $P'Q'$ fixed, let the element be further stretched, without altering the curvature, so that its length ds becomes ds', then $dp = \dfrac{ds' - ds}{ds_1}$. The work done by the tension T at the end Q' is $-T(ds' - ds)$, and that done by the couple at Q' is $-L\dfrac{ds' - ds}{\rho}$. We therefore have

$$-\left(T + \frac{L}{\rho}\right) = \frac{dW}{dp} \quad \text{.....................................(5)}.$$

Next let the element, without altering its length, receive an increase of curvature so that the radius of curvature is changed from ρ to ρ'; then $\dfrac{dq}{a} = \dfrac{1}{\rho'} - \dfrac{1}{\rho}$. The

tension at Q' does no work, while the work of the couple L at Q' is $-L\left(\frac{1}{\rho'}-\frac{1}{\rho}\right)ds$.

In this way we find
$$-\frac{L}{a}=\frac{dW}{dq}\frac{1}{1+p} \quad\text{.............................}(6).$$
These expressions give for a slightly extensible and flexible rod
$$L=\frac{Kq}{a}, \qquad T=Hp-\frac{K}{a^2}q \quad\text{.........................}(7).$$

The equations of equilibrium found in Arts. 10 and 11 when joined to (7) supply five equations from which L, T, U, u, ϕ may be found.

39. Ex. One end of a heavy, slightly flexible wire, in the form of a circular quadrant, is fixed into a vertical wall, so that the plane of the wire is vertical and the tangent at the fixed end horizontal. Assuming that the change of curvature at any point is proportional to the moment of the bending couple there, prove that the horizontal deflection at the free end is $\frac{\pi}{8}\frac{wa^4}{E}$, where E is the flexural rigidity, w the weight of a unit of length, and a the radius of the circle. [Trin. Coll. 1892.]

Let A be the free end of the rod, B the end fixed into the wall, O the centre. Taking moments about any point P for the side PA, Art. 10, we arrive at
$$\frac{E}{a^3w}\left(u+\frac{d^2u}{d\theta^2}\right)=-\sin\theta+\theta\cos\theta,$$
where $AOP=\theta$, and $OP=a(1+u)$. The constants of integration are determined from the conditions that u and $du/d\theta$ vanish at B, and the deflection required is the value of au when $\theta=0$.

40. *To find the work when a thin rod, whose central line in the natural state is a circle of radius a, is stretched and bent so that the central line becomes a circle of radius ρ, by a method analogous to that used in Art.* 33 *for a straight rod.*

The figure of Art. 33 may be used in what follows, except that the lines OM, AB, CD must be supposed to be small arcs of circles.

Let OM be an element of the central line of the unstrained solid, $O'M'$ the same element when the rod is deformed. Let the tangents to OM, $O'M'$ be the axes of x and x', and let the planes of xz, $x'z'$ be the planes of the circles. Let QR be any filament parallel to OM, $Q'R'$ its position in the strained rod. Let y, z ; y', z' be the coordinates of Q, R; Q', R', each referred to its own set of axes.

If ds_1, ds be the lengths of OM, $O'M'$ and $1+p$ stand for ds/ds_1 as before, the tension of $O'M'$ per unit of area is Ep. If $d\sigma_1$, $d\sigma$ be the lengths of QR, $Q'R'$, we have
$$d\sigma_1=ds_1\left(1-\frac{z}{a}\right), \qquad d\sigma=ds\left(1-\frac{z'}{\rho}\right) \quad\text{.................}(1),$$
and the resultant tension of all the fibres which cross the area $dydz$ is therefore
$$Edydz\left(\frac{d\sigma}{d\sigma_1}-1\right) \quad\text{.................................}(2).$$
The work done by this tension when the filament is pulled from its unstretched length $d\sigma_1$ to the length $d\sigma$ is
$$-\tfrac{1}{2}Edydz\left(\frac{d\sigma}{d\sigma_1}-1\right)^2 d\sigma_1 \quad\text{..............................}(3).$$

The difference $z'-z$ is a small fraction of z; for a straight rod it has been shown to be of the order z^2/ρ, Art. 34. As a first approximation we take $z'=z$. Substituting for ds/ds_1 and for $1/\rho$ their values $1+p$ and $(1+q)/a$, and neglecting all powers of z/a above the second, we find that the work is
$$-\tfrac{1}{2}Edydz\left[p^2-\{p^2+2pq(1+p)\}\frac{z}{a}+q^2(1+p)^2\left(\frac{z}{a}\right)^2\right]ds_1 \quad\text{............}(4).$$

Integrating this over the area ω of the section, and remembering that O is the centre of gravity of the area, we have for the whole work

$$Wds_1 = -\tfrac{1}{2}E\omega\left[p^2 + \frac{k^2}{a^2}q^2(1+p)^2\right]ds_1 \quad\text{.......................(5)};$$

when the higher powers of p, q are rejected this becomes

$$Wds_1 = -\tfrac{1}{2}E\omega\left[p^2 + \frac{k^2}{a^2}q^2\right]ds_1 \quad\text{.........................(6)}.$$

41. In the same way we find that the tension of the fibres which cross the area $dydz$ is

$$Edydz\left[p - q(1+p)\left\{\frac{z}{a} + \left(\frac{z}{a}\right)^2\right\}\right] \quad\text{.......................(7)}.$$

Remembering that O is the centre of gravity of the section, we find by an obvious integration that the resultant tension T and the resultant couple L are given by

$$T = E\omega\left\{p - \frac{k^2}{a^2}q(1+p)\right\}, \qquad L = E\omega a\frac{k^2}{a^2}q(1+p) \quad\text{...........(8)}.$$

These reduce to the forms given in Art. 38 when the product pq is neglected.

42. If we examine the expressions for the work, tension and couple given by equations (6) and (8) of Arts. 40, 41 we see that they contain two constants of elasticity, viz. $E\omega$ and $E\omega k^2$. These were represented by the letters H, K in the corresponding expressions in Art. 38.

When the rod is such that the constant of elasticity $E\omega$ is infinite or very great, a small change in the proportional extension p alters the product $E\omega p$ very considerably. Unless, therefore, the circumstances of the problem allow the tension to be infinite or very great, p must be very nearly equal to zero. It follows that in all the *geometrical relations* of the figure we may regard p as equal to zero. At the same time the product $E\omega p$ which occurs in the tension is not to be regarded as zero, but as a quantity analogous to the singular form $\infty \cdot 0$. If the tension is finite, the term Ep^2 which occurs in the work is zero.

Since the other constant of elasticity, viz. $E\omega k^2/a^2$, is not necessarily large in thin rods, it does not follow that, because $E\omega$ is large, q must be small.

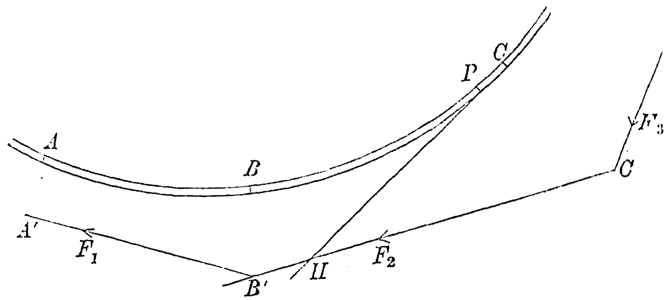
Rods in which $E\omega$ is very great are said to be inextensible. Such rods may be bent, and the bending couple is proportional to the change of curvature.

43. **Very flexible rod.** When the flexibility of the rod is such that it may be made to pass through several small rings not nearly in one straight line the integrations of the differential equation become more intricate. To simplify the problem we suppose that though weights may be attached to any points, the rod itself is without weight.

Let A, B, C &c. be a series of small smooth rings through which the rod is passed. Let the stress couple at A be L_1, and let T_1, U_1 be the tension and shear at the same point. Let L_2, T_2, U_2 be the corresponding stresses at B and so on.

The stress L_1, T_1, U_1 acting at A may be reduced to a single resultant F_1 acting along some straight line $A'B'$, whose position is found in Vol. I. Art. 118. If P be any point between the rings A and B, the stress at P must be equivalent to the same force, for otherwise the portion AP of the light rod would not be in equilibrium. In the same way the stress at every point of the rod between the rings B and C is equivalent to a single resultant F_2 acting along some other straight line $B'C'$, and so on for every portion of the rod lying between contiguous rings. These straight lines may be called *the lines of pressure.* We shall suppose

the forces F_1, F_2, &c. when positive to be pulling forces, so that for instance the action of AP on PB is equivalent to the force F_1 acting in the direction $B'A'$.



The stress forces at points one on each side of any ring, as B, being F_1 and F_2, it follows that the pressure on the ring B is the resultant of F_1 acting along $B'A'$ and F_2 reversed and thus made to act along $B'C'$. The pressure at B therefore acts along $B'B$, and this line is a normal to the rod at B.

Let us consider the equilibrium of any portion BP of the rod, where P is a point between B and C. Let ψ be the angle the tangent PH makes with $B'C'$, and let $B'C'$ be the axis of ξ. Let η be the perpendicular distance of P from that axis. Let L, T, U be the stress couple, tension and shear at P. Then

$$T=F_2\cos\psi,\qquad U=-F_2\sin\psi,\qquad L=F_2\eta \quad\text{.....................(1).}$$

Taking moments about P for the portion BP we have

$$\frac{K}{\rho}=K\frac{d\psi}{ds}=F_2\eta \quad\text{.....................................(2).}$$

Multiply both sides by $\sin\psi=d\eta/ds$ and integrating, we find $-2K\cos\psi=F_2\eta^2+C$. This result may be written in the form

$$2KF_2\cos\psi+F_2^2\eta^2=I \quad\text{.................................(3),}$$

where I is a constant for the portion BP of the rod. We notice that in this equation $F_2\cos\psi$ is the tension and $F_2\eta$ the stress couple at the point P.

A similar equation holds for each portion of the rod which lies between contiguous rings. If P move along the rod and pass through the ring C, the tension and stress couple undergo no sudden changes of value, though the shear is altered discontinuously. It follows that $F_2\cos\psi$ and $F_2\eta$ are the same on both sides of C and that therefore I is the same for both portions of the rod. *The constant I has therefore the same value throughout the whole length of the rod.*

If one extremity of the rod is free, let A be the ring nearest the free end. The tension and the stress couple at A are therefore zero; hence, by equation (3) the value of I is zero. In this case, since the stress at A is reduced to the shear only, the line of pressure between the rings A and B is the normal at A.

Since $\rho\cos\psi=d\xi/d\psi$, we have

$$\frac{d\xi}{d\psi}=\frac{K\cos\psi}{F_2\eta}=\frac{K\cos\psi}{(I-2KF_2\cos\psi)^{\frac{1}{2}}} \quad\text{.........................(4),}$$

where ξ is measured positively opposite to the direction of F_2. Putting $\psi=\pi-2\theta$, we reduce this to the difference of two elliptic integrals,

$$F_2\xi=i\int(1-c^2\sin^2\theta)^{\frac{1}{2}}\,d\theta-\frac{I}{i}\int\frac{d\theta}{(1-c^2\sin^2\theta)^{\frac{1}{2}}},$$

where $i^2=I+2KF_2$ and $c^2i^2=4KF_2$.

44. To show that these results supply a sufficient number of equations, let us suppose, as an example, that both ends of the rod are free and that it has been made to pass through five small rings at A, B, C, D, E.

Beginning at the ring A, the line of pressure $A'B'$ is the normal at A; let θ be the angle it makes with any fixed straight line in the plane of the rings. Taking AB' as the axis of ξ and A as origin, the coordinates of B, viz. ξ, η, are known functions of θ. The equations (3) and (4) give ξ, η, in terms of ψ and F_1, the constant in (4) being determined from the condition that when $\xi=0$ the value of ψ is known, viz. in this case ψ is a right angle. Equating these two values of ξ and η we have two equations to determine F_1 and the value of ψ at B. The tangent at B having been found, the normal BB' can be drawn and the position of B determined.

If $\pi-\beta$ be the angle $A'B'C'$, we have $F_2 \cos\beta=F_1$. When therefore we repeat the process just described and take $B'C'$ as a second axis of ξ and the foot of the perpendicular from B on $B'C'$ as the origin, with the object of finding F_2 and the value of ψ at C, we really have sufficient equations to find β also.

In the same way we next take $C'D'$ as the axis of ξ and finally $D'E'$. But since this last line of pressure must be the normal at E, the value of ψ at E must be a right angle. This supplies a final equation from which θ may be found.

Ex. A light rod DE is made to pass through two small rings A, C in the same horizontal line at a distance apart equal to $2b$, and has a weight W applied at a point B so that the vertical through B bisects AC at right angles. If 2ϕ be the angle between the normals at A and C prove that

$$2\cos\phi\,(\cot\phi)^{\frac{1}{2}}+(\cos\phi)^{\frac{1}{2}}\int_0^{\phi}(\sin\phi)^{\frac{1}{2}}d\phi=b\operatorname{cosec}\phi\left(\frac{W}{K}\right)^{\frac{1}{2}}.$$

On rods in three dimensions.

45. **Measures of Twist.** Let PK be a normal to the central line of an elastic rod at any point P, and let K lie on the outer boundary of the rod, the portion PK is called a *transverse* of the rod. This name is due to Thomson and Tait.

Let P, P', P'' &c. be a series of adjacent points on the central line of the unstrained rod, and let each of the arcs PP', $P'P''$ &c. be infinitesimal. Any transverse PK having been drawn at the first of these points, let the plane KPP' intersect the normal plane at P' in a second transverse $P'K'$. Let the plane $K'P'P''$ intersect the normal plane at P'' in a third transverse, and so on. We thus obtain a series of transverses, any consecutive two of which lie in a tangent plane to the central line.

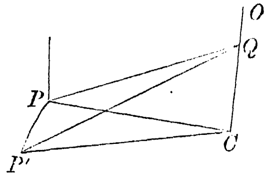
If the rod when unstrained is straight and cylindrical it is obvious that all the transverses thus drawn lie in a plane passing through the central line. It is also clear that the extremities K, K' &c. of the transverses then trace out a straight line on the surface of the rod parallel to the central line.

Let these transverses be fixed in the material of the rod and move with it when the rod is strained. The normal section at P of the rod being fixed, let the elements lying between the normal planes at P, P', P'' &c. be twisted round the tangents $PP', P'P''$ &c. respectively, so that the points K, K', K'' &c. trace out a spiral line on the outer boundary of the rod. The twist of the elementary portion of the rod which lies between the normal planes at P, P' is measured by the infinitesimal angle which the transverse $P'K'$ makes with the plane KPP', or, what is ultimately the same, by the angle which the planes KPP', $PP'K'$ make with each other. If the arc PP' of the central line be ds, and if the angle which the planes KPP', $PP'K'$ make with each other be $d\chi$, the ratio $d\chi/ds$ represents the twist of the portion ds of the rod referred to a unit of length, and is usually called *the twist at P*.

It is sometimes useful to so choose the transverses PK, $P'K'$ &c. in the unstrained rod that the angle which the planes KPP', $PP'K'$ make with each other has any convenient value. Let $d\chi_1$ be this angle and let $d\chi_1 = \tau_1 ds$, then τ_1 is an arbitrary function of the arc s. If $d\chi$ or τds be the corresponding angle in the strained rod, the twist is measured by $\tau - \tau_1$.

46. **Resolved Curvature.** Let a straight rod be strained by bending, so that the central line takes the form of a curve of double curvature. If $d\epsilon$ be the angle between the normal planes to the central axis at P, P', the curvature at P is measured by the ratio $d\epsilon/ds$, and the central line is said to be curved in the osculating plane.

It is sometimes more convenient to resolve the curvature in two directions at right angles. Let the normal planes at P, P' intersect each other in a straight line CO, then CO intersects the osculating plane at right angles in some point C. Since PC, $P'C$ are two consecutive normals lying in the osculating plane, the point C is the centre of the circle of curvature; let $CP = \rho$. Let us now draw a plane through the tangent PP' to the central line making an arbitrary angle ϕ with the osculating plane, and let this plane cut CO in Q. Then since PQ, $P'Q$ are two consecutive normals to the central line, the point Q is the centre of a circle of curvature



drawn in the plane QPP'. If the radius PQ of this circle be R, we have from the right-angled triangle QCP

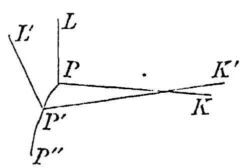
$$\frac{1}{R}=\frac{1}{\rho}\cos\phi.$$

It follows that *the curvature in a plane drawn through the tangent may be deduced from the curvature in the osculating plane by the same rule that we use in statics to resolve a force.*

47. Let us draw two planes through the tangent at P to the central line, and let these be at right angles to each other. Let the resolved curvatures of the central line in these planes be called κ and λ. Then the curvature in the osculating plane is $\sqrt{(\kappa^2+\lambda^2)}$, and the tangent of the angle the osculating plane makes with the first of these planes is λ/κ.

These two planes intersect the normal plane at P to the central line in two straight lines at right angles. Let these be PK, PL, the straight line PK being perpendicular to the plane in which the resolved curvature is κ.

The three straight lines PK, PL, PP' thus form a convenient system of orthogonal axes to which we may refer that part of the rod which lies in the immediate neighbourhood of P. The resolved curvatures of the central line in the planes perpendicular to PK, PL, being κ, λ and the twist about PP' being τ, it follows that in passing from the point P to P' the three axes are screwed into positions $P'K'$, $P'L'$, $P'P''$ by a combination (1) of the rotations κds, λds, τds about the axes PK, PL, PP' and (2) of a translation of the origin P along the tangent to P'. It should be noticed that each of the three quantities κ, λ, τ is of -1 dimension as regards space.



The quantities κ, λ are the resolved curvatures of the strained rod and are the same as the resolved bendings produced by the forces only when the unstrained rod is straight. To find the bending produced by the external forces when the unstrained rod is itself curved we must subtract from κ, λ the resolved curvatures of the unstrained rod.

48. Since κds, λds, τds are rotations about the axes of reference, we know by the parallelogram of angular velocities that they may be resolved about other axes by the parallelogram

law. If then we wish to refer the strains to a different set of axes, say PK_1, PL_1, PT_1, we change κ, λ, τ into κ_1, λ_1, τ_1 by the usual formulæ for the transformation of coordinates or for the resolution of forces. In this way we may refer the bending and twist in the neighbourhood of P to any arbitrary system of axes having the origin at P. These generalized axes may be screwed from their positions at the origin P to those at P' by the three rotations $\kappa_1 ds$, $\lambda_1 ds$, $\tau_1 ds$ and the translation ds along the tangent.

49. In many of the applications of analytical geometry to physical problems it is found advantageous to make the coordinate axes moveable, so that they may always be in the most convenient position. Thus if a point travel along the strained rod and successively occupy the positions P, P', P'', &c., the axes change their directions in space. To specify the motion of these axes we may either use a second system of axes fixed in space or we may refer the motion to the moving axes themselves in the manner above described. The first method requires the use of the formulæ of transformation of axes which are often complicated, in the second we avoid the introduction of a second system of axes. Moving axes are of great importance in Dynamics and are also of much use in discussing the geometrical properties of curves and surfaces. For these applications the reader is referred to the second volume of the Author's treatise on Rigid Dynamics.

50. Ex. 1. A straight line is marked on the surface of a thin unstrained cylindrical rod, parallel to the central line. If the rod is bent along any curve on a spherical surface so that the marked line is laid in contact with the spherical surface, show that the twist is zero.

If the rod is laid on a cylindrical surface so that the marked line is in contact with the cylinder, show that the twist is $\sin\alpha\cos\alpha/a$, where a is the radius of the cylinder and α is the angle the rod makes with the axis of the cylinder. Both these results are given by Thomson and Tait, Art. 126.

If P, P' be two consecutive points on the central line, the transverses PK, $P'K'$ are normals to the surface. The first result follows, because the transverses pass through the centre of the sphere, so that the angle between the planes KPP', $PP'K'$ is zero. Since the radius of curvature at any point of a helix lies on the normal to the cylinder on which the helix is drawn, the second result follows from the ordinary expression for the radius of geometrical torsion.

Ex. 2. A straight thin rod has a straight line marked along one side. If the rod is bent and laid on a surface so that this line lies in contact with a geodesic, show that the twist at any point P is $\Delta\sin\theta\cos\theta$, where Δ is the difference of the curvatures of the principal sections of the surface at P and θ is the angle the rod makes with either line of curvature.

51. **Relations of stress to strain.** Let P be any point on the central line; the mutual action of the parts of the rod on each side of the normal section at P can be reduced to a force and a couple with any convenient point of that section as base.

Let three rectangular axes be taken at the point P to which we may refer the strains and stresses in the neighbouring portion

of the rod. Let K, L, T be the components of the stress couple about these axes. If the unstrained rod is straight, let κ, λ, τ be the resolved parts of the curvature and twist about the axes; if the unstrained rod is itself curved, then κ, λ, τ represent the changes in the curvature and twist produced by the external forces.

We shall now assume the two following principles*.

(1) that the changes in the twist and curvature of the rod in the neighbourhood of P are independent of the force and are functions only of the couple.

(2) that the couples K, L, T are linear functions of the strains κ, λ, τ.

These assumptions are necessary because we do not in this place enter into the theory of elasticity.

If we suppose, as usual, that the strains are so small that we may neglect all powers but the lowest which enter into the equations, the second principle is equivalent to the assumption that when K, L, T are expanded in powers of κ, λ, τ the lowest powers in the series are the first.

52. Since the three couples K, L, T are each expressed in terms of κ, λ, τ by a different linear equation, it might be supposed that we shall have to deal with nine constants. But if the elastic forces form a conservative system we may reduce these to six by using the work function.

Let Wds be the work function of an element of the rod bounded by the normal sections at P, P'. Supposing the end P fixed, let one strain, say λ, become $\lambda + d\lambda$, the other two remaining unaltered. Since the element of the rod has been rotated about the axis of the couple L through an angle equal to $d\lambda \, . \, ds$, the work done by the couple L is $Ld\lambda ds$, while that done by each of the couples K and T is zero. We therefore have $ds\,dW = Ld\lambda ds$. Similar expressions hold when K and T are increased by $d\kappa$ and $d\tau$, so that in general

$$K = dW/d\kappa, \qquad L = dW/d\lambda, \qquad T = dW/d\tau.$$

Since K, L, T are linear functions of κ, λ, τ it follows that W is a quadratic function of κ, λ, τ, i.e.

$$W = \tfrac{1}{2}(A\kappa^2 + B\lambda^2 + C\tau^2 + 2a\lambda\tau + 2b\tau\kappa + 2c\kappa\lambda).$$

$$\therefore\ K = A\kappa + c\lambda + b\tau,$$
$$L = c\kappa + B\lambda + a\tau,$$
$$T = b\kappa + a\lambda + c\tau.$$

* See Thomson and Tait, 1883, Art. 591.

53. We have already seen that if we refer the strains to another set of axes the quantities κ, λ, τ are changed by the ordinary formulæ for transformation of coordinates, Art. 48. Since a homogeneous quadratic expression can always be cleared of the terms containing the products of the variables, it follows that by a proper choice of the axes of reference the expressions for W, and therefore those for K, L, T may be reduced to the simplified forms

$$W = \tfrac{1}{2}(A_1\kappa_1^2 + B_1\lambda_1^2 + C_1\tau_1^2)$$

$$K_1 = A_1\kappa_1, \qquad L_1 = B_1\lambda_1, \qquad T_1 = C_1\lambda_1.$$

These axes are called *the principal axes of stress* and the constants A_1, B_1, C_1, are *the principal flexure and torsion rigidities.*

In what follows it will generally be assumed that the tangent to the central line at P is one principal axis of stress at P; this is of course the axis of torsion. If also the constants of rigidity for the other two principal axes are equal, we have

$$W = \tfrac{1}{2}A\,(\kappa^2 + \lambda^2) + \tfrac{1}{2}C\tau^2$$

where the suffixes have been dropped as being no longer required.

The expression for the work is not complete if the rod is extensible, for we have not yet taken account of the extension or stretching, of the element PP' of the rod. This additional term is given in Vol. I. on the supposition that the torsion obeys Hooke's law. It will not be required in the problems considered in this chapter.

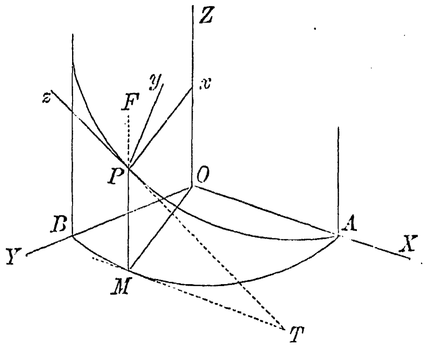
54. **Helical twisted rods.** *A uniform thin rod, naturally straight, whose principal stress axes at any point are the central line and any two perpendicular axes, is bent into the form of a helix of given angle and receives at the same time a given uniform twist. It is required to find the force and couple which must be applied at one extremity, the other being fixed, that the rod may retain the given strains.*

Let APQ be an arc of the helix, A the fixed extremity, Q the terminal at which the forces are applied. Let AMB be a circular section of the cylinder on which the helix lies, OZ the axis of the cylinder.

The action of the portion AP of the helix on the portion PQ consists of a force and a couple. From the uniformity of the figure it is clear that the force and couple must be the same in magnitude wherever the point P is taken on the helix, and that their direc-

tion and axis respectively must make the same angles with the principal axes of the curve at P.

The stress force at P may be resolved into two components, one acting along the generating line MPF of the cylinder and the other acting parallel to the plane XY. The latter, if it is not zero, must be in equilibrium with the component at Q parallel to the same plane. This however is impossible, because as P is moved along the helix the direction of the component at P makes always the same angles with the principal axes at P and is therefore changed, while that of the component at Q remains unaltered. Both these components must therefore be zero. It follows that the resultant stress force at any point P must act along the generator through that point.



Let R be the stress force at any point of the wire and let it be estimated as positive in the direction FPM. The force R may be transferred to the axis OZ of the cylinder by introducing the couple Ra acting in the plane $OZFM$. The force R thus becomes independent of the position of P.

Let us now turn our attention to the stress couples at P. Let Px be drawn perpendicular to the axis of the cylinder and let TPz be a tangent at P to the helix. Then by the known properties of the curve, the plane TPx is the osculating plane at P. Let Py be the binormal. If $\rho = 1/\kappa$ be the radius of curvature of the helix, the strains round Py and Pz are respectively κ and τ, each being measured in the positive direction round the axes i.e. from z to x and x to y. If A and C are the constants of flexure and torsion, the corresponding stress couples are $A\kappa$ and $C\tau$. These couples may be resolved into two components, one having the generator PF for axis and the other having its axis parallel to the plane of XY.

The latter, together with the couple Ra, must be in equilibrium with the corresponding component at the terminal Q. But since the axis of the couple at P always makes the same

angle with OM, its direction is altered when the point P is moved along the helix, while that of the couple at Q is fixed. Equilibrium therefore cannot exist in all positions of P unless each of these components is zero. The axis of the stress couple at P must therefore be parallel to the axis of the cylinder.

It follows from this reasoning that if a thin straight rod, with one end fixed, have any given twist and be bent into the form of a helix, it may be maintained in that position by the action of a force and a couple at one terminal. These when reduced to a wrench will consist of a force R acting along the axis of the cylinder together with some couple G whose plane is perpendicular to that axis.

Taking moments about an axis perpendicular to the plane MPx, we have
$$Ra = -A\kappa \sin\alpha + C\tau \cos\alpha \quad\text{.................(1)}.$$
Taking moments about the generator of the cylinder
$$G = A\kappa \cos\alpha + C\tau \sin\alpha \quad\text{...................(2)}.$$
These equations determine R and G when the angle α of the spiral, its curvature κ and the twist τ of the material are known.

55. **Spiral Springs.** *A thin rod or wire whose natural form is a given helix and whose principal axes of stress at any point are the tangent to the central line and any two perpendicular axes is bent into the form of another given helix. It is required to find the forces and couples which must be applied at one end, the other being fixed, that the rod may retain the given form.*

Let a_1, a be the radii of the cylinders on which the unstrained and strained helices lie; α_1, α the angles of the helices. Let the axes of the two cylinders be coincident and let it be taken as the axis of Z, the plane of XY being perpendicular to it.

Let P, P' &c. be a series of consecutive points on the central line of the unstrained rod and let $P\xi$, $P'\xi'$ &c. be the principal normals at these points. The angle between the consecutive planes $\xi PP'$, $PP'\xi'$ is $ds \sin\alpha_1 \cos\alpha_1/a_1$ where ds is the arc PP'. Let $P\eta$, $P'\eta'$ be the binormals at the same points, then the curvature of the unstrained rod, measured, as in Art. 47, round the binormal, is $ds \cos^2\alpha_1/a_1$. Let $P\zeta$, $P'\zeta'$ be the tangents to the helix taken positively in the direction in which s is measured.

Let the axes of ξ, η be fixed in the material of the rod and be the transverses of reference. When the rod is strained, let Px, Py, Pz be the principal normal, binormal and tangent at P, then $P\zeta$

coincides with Pz, and $P\xi$ makes some angle ϕ with Px. The figure of Art. 54 may be used to represent the strained position of the rod, the axes $P\xi$, $P\eta$, $P\zeta$ are not drawn but may easily be supplied by the description just given.

The stress at the point P of the strained rod consists of (1) a force which we may suppose to be resolved into two components, one along the generating line of the cylinder and the other parallel to the plane of XY. (2) A couple $C(\tau - \tau_1)$, whose axis is the tangent Pz and two couples $A\kappa$, and $-A\kappa_1$, whose axes are Py and $P\eta$ respectively; where

$$\tau_1 = \frac{\sin\alpha_1 \cos\alpha_1}{a_1}, \quad \kappa_1 = \frac{\cos^2\alpha_1}{a_1}, \quad \kappa = \frac{\cos^2\alpha}{a},$$

and τds is the elementary angle between the planes $\xi PP'$, $PP'\xi'$ in the strained rod.

Examining first the stress force, we find, as in Art. 54, that the component parallel to the plane of XY is zero. The stress force at every point P therefore acts along the generating line of the cylinder; let this force be R, and let it be transferred to the axis of the cylinder by introducing a couple Ra.

Taking next the stress couples, we find by the same reasoning that the component about any axis parallel to the plane of XY is zero. Let us first equate to zero the moment about Px; since Px is perpendicular to Py, Pz and the axis of the couple Ra, and makes with $P\eta$ an angle $\frac{1}{2}\pi + \phi$ we have $\kappa_1 \sin\phi = 0$. Since κ_1 is not zero, as in Art. 51, it follows that $\phi = 0$. The axes $P\xi$ and Px therefore coincide and the couples $A\kappa$ and $-A\kappa_1$ have a common axis Py, viz. the binormal of the strained helix. The angle τds is also equal to the angle between the consecutive osculating planes to the strained helix, i.e. $\tau = \sin\alpha\cos\alpha/a$.

Equating to zero the moment about a perpendicular to the plane passing through Px and the generator of the cylinder, we have

$$Ra = -A\sin\alpha(\kappa - \kappa_1) + C\cos\alpha(\tau - \tau_1) \quad\text{.............(1).}$$

Equating the moment about a generator to the corresponding moment at the terminal we have

$$G = A\cos\alpha(\kappa - \kappa_1) + C\sin\alpha(\tau - \tau_1) \quad\text{.............(2).}$$

The curvatures and torsions are

$$\kappa_1 = \frac{\cos^2\alpha_1}{a_1}, \quad \kappa = \frac{\cos^2\alpha}{a}, \quad \tau_1 = \frac{\sin\alpha_1\cos\alpha_1}{a_1}, \quad \tau = \frac{\sin\alpha\cos\alpha}{a}.$$

56. If the spiral spring have a great many turns so that α_1

and α are both small, we have when the squares of α_1, α are neglected

$$Ra = -A\alpha\left(\frac{1}{a} - \frac{1}{a_1}\right) + C\left(\frac{\alpha}{a} - \frac{\alpha_1}{a_1}\right)$$

$$G = A\left(\frac{1}{a} - \frac{1}{a_1}\right).$$

If there be no couple G but only a force at each end pulling the spiral out, we deduce from these equations that $a = a_1$, so that the spring occupies a cylinder of the same radius as before the strain.

We also have
$$Ra = C\frac{\alpha - \alpha_1}{a},$$
which is independent of the constant of flexure. It appears therefore that *the spring resists the pulling out chiefly by its torsion.* This result is ascribed to Binet in 1815 by Saint-Venant and by Thomson and Tait.

Let l be the length of the spiral spring, h the elongation of its axis produced by the force R; then

$$l\sin\alpha - l\sin\alpha_1 = h.$$

Rejecting as before the squares of α_1 and α we find that $R = C\,.\,\dfrac{h}{la^2}$. This expression determines the force required to produce a given elongation in a given spring of small angle.

57. **Equations of Equilibrium.** *To form the general equations of equilibrium* in three dimensions of a strained rod.*

Let P, P' &c. be a series of consecutive points on the central line of the unstrained rod. Let a series of transverses PK, $P'K'$ &c. be drawn such that the angle of twist τ_1 is either zero or some arbitrary function of the arc s. Taking the transverse PL perpendicular to PK, let the resolved curvatures about these lines be λ_1 and κ_1. If these transverses are the principal flexure and torsion axes at each point of the rod they form a convenient system of coordinate axes. If not let some other system of axes, $P\xi$, $P\eta$, $P\zeta$ be chosen which are connected with the transverses PK, $P'K'$ &c. in a known manner. Let θ_1, θ_2, θ_3 be the resolved parts of κ_1, λ_1, τ_1 about the axes $P\xi$, $P\eta$, $P\zeta$. Then, as explained in Art. 48, the axes at P may be brought into positions parallel to those at P' by rotations $\theta_1 ds$, $\theta_2 ds$, $\theta_3 ds$ about themselves.

* The general equations of a rod in Cartesian coordinates may be found in the *Treatise on Natural Philosophy* by Thomson and Tait, 1879. The intrinsic equations, or those referred to moving axes, are given in the *Treatise on Statics* by Minchin, 1889.

When the rod is strained the axes $P\xi$, $P\eta$, $P\zeta$ will move with the material of the rod and assume new positions in space. Let these be Px, Py, Pz. Let $\omega_1 ds$, $\omega_2 ds$, $\omega_3 ds$ be the rotations by which the axes at P in the strained rod are brought into positions parallel to those at P'. The differences $(\omega_1 - \theta_1)\,ds$, $(\omega_2 - \theta_2)\,ds$, $(\omega_3 - \theta_3)\,ds$ may be used to measure the strains produced by the external forces.

Let R_1, R_2, R_3; L_1, L_2, L_3 be the stress forces and couples which act at P on the element PP' in front of P and let them be estimated as positive when they act in the negative directions of the axes at P. Then $R_1 + dR_1$ &c., $L_1 + dL_1$ &c., are the corresponding forces and couples at P' and act on the element PP', behind P', in the positive directions of the axes at P'. Besides these the element is acted on by the impressed forces $F_1 ds$, $F_2 ds$, $F_3 ds$ and the impressed couples (if any) $G_1 ds$, $G_2 ds$, $G_3 ds$.

Since R_1, R_2, R_3 are the components of a vector, viz. the stress force at P, the differences of the resolved parts at P and P' along the same set of axes are given by the rule for resolving vectors*, we therefore have

$$\frac{dR_1}{ds} - \omega_3 R_2 + \omega_2 R_3 + F_1 = 0$$

* The following proof of the rule is the same as that given in the second volume of the Author's Rigid Dynamics.

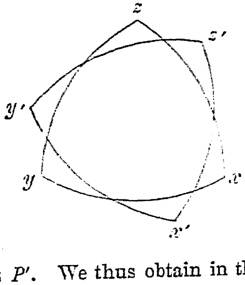
Describe a sphere of unit radius whose centre is at P and let the axes Px, Py, Pz cut its surface in x, y, z. Let parallels to the corresponding axes at P' drawn through P cut the surface in x', y', z'. Thus we have two spherical triangles xyz, $x'y'z'$, all whose sides are quadrants. Also x, y, z are brought into coincidence with x', y', z' by the combined effect of the rotations $\omega_1 ds$, $\omega_2 ds$, $\omega_3 ds$ about Px, Py, Pz respectively.

Let U, V, W be the components of the vector at P in the directions of the axes x, y, z, $U + dU$, &c. the components of the vector at P' along the axes x', y', z'. The difference of the resolved parts along the axis of x is then

$$(U + dU)\cos xx' + (V + dV)\cos xy' + (W + dW)\cos xz' - U.$$

The rotations about Px, Py cannot alter the arc xy, but the rotation about Pz will move y' away from x by the arc $\omega_3 ds$. In the same way the rotations about Px and Pz cannot alter the arc xz, but the rotation about Py will move z' towards x by the arc $\omega_2 ds$. Therefore

$$xy' = xy + \omega_3 ds, \qquad xz' = xz - \omega_2 ds.$$



Also the cosine of the arc xx' differs from unity by the square of a small quantity. Substituting we find that the difference of the resolved parts along the axis of x is $\quad dU - V\omega_3 ds + W\omega_2 ds.$

If U, V, W stand for R_1, R_2, R_3 we join to this the force $F_1 ds$, equating the result to zero and dividing by ds, we obtain the first of the six equations. If U, V, W stand for L_1, L_2, L_3 we add the couple $G_1 ds$ and the moments of the forces $R_1 + dR_1$ &c. acting at P'. We thus obtain in the same way the fourth of the six equations.

$$\frac{dR_2}{ds} - \omega_1 R_3 + \omega_3 R_1 + F_2 = 0$$

$$\frac{dR_3}{ds} - \omega_2 R_1 + \omega_1 R_2 + F_3 = 0.$$

In the same way since L_1, L_2, L_3 are the components of a vector,

we have
$$\frac{dL_1}{ds} - \omega_3 L_2 + \omega_2 L_3 + G_1 + \mu R_3 - \nu R_2 = 0$$

$$\frac{dL_2}{ds} - \omega_1 L_3 + \omega_3 L_1 + G_2 + \nu R_1 - \lambda R_3 = 0$$

$$\frac{dL_3}{ds} - \omega_2 L_1 + \omega_1 L_2 + G_3 + \lambda R_2 - \mu R_1 = 0,$$

where λ, μ, ν are the direction cosines of the arc PP' referred to the axes at P.

The relations between the couples L_1, &c., and the strains $\omega_1 - \theta_1$, &c., may be deduced from the expression for the work W given in Art. 52, by writing $\omega_1 - \theta_1$, &c. for κ, τ, λ. Supposing for the sake of brevity that the axes are the principal flexure and torsion axes, we have

$$L_1 = A(\omega_1 - \theta_1), \quad L_2 = B(\omega_2 - \theta_2), \quad L_3 = C(\omega_3 - \theta_3).$$

If the axes are the tangent at P to the central line and two perpendicular axes, we have $\lambda = 0$, $\mu = 0$ and $\nu = 1$; but in all cases λ, μ, ν are known from the given conditions of the rod.

We thus have nine equations to determine the quantities R_1, R_2, R_3; L_1, L_2, L_3; ω_1, ω_2, ω_3. If the rod is extensible there will be another equation supplied by Hooke's law.

ASTATICS.

On Astatic Couples.

1. THE conditions of astatic equilibrium in two dimensions have already been investigated in the first volume of this treatise. We have now to consider what other conditions are necessary when the body is displaced in any manner in three dimensions*.

2. We shall suppose, as before, that each force acting on the body retains the same direction in space, the same magnitude and continues to act at the same point of the body, for all displacements.

The forces of a couple remain parallel, equal, and unaltered in magnitude as the body is moved, but the length of the arm is not necessarily the same. Let A, B be the points of application of the forces, then the distance AB is unaltered, and is called the *astatic arm of the couple.* If in any position of the body the

* The subject of Astatics appears to have been first studied by Moebius, who published his results in his *Lehrbuch der Statik*, 1837. Moigno also, in his *Statique*, has discussed the subject at great length. Minding in the fifteenth volume of Crelle's *Journal* gave the theorem that, whenever the body is so placed that the forces admit of a single resultant, that resultant intersects two conics fixed in the body. Many proofs have been given of this curious theorem; we may mention that by Darboux, Tait's proof by quaternions modified by Minchin; Larmor's proof with the use of the six coordinates of a line.

Darboux published in the *Mémoires de la Société des Sciences physique et Naturelles de Bourdeaux*, t. II. [2e Série), 2e Cahier, a very long paper on this subject. In contradiction to Moebius, he showed that when one point of a body is fixed there are in general four positions, and only four, in which the body can be placed so that the forces are in equilibrium. These he called the initial positions of the body. His investigation is rather long and a different proof is given here. He also introduced the idea of a central ellipsoid analogous to the momental ellipsoid used in discussing moments of inertia. This result is given in Art. 14 of the text, and the general lines of his argument have been followed in that article. By the use of this ellipsoid he gave a geometrical turn to the proof of Minding's theorem, but it remained rather complicated. Extending the theory by considering all positions of the body, he showed that Poinsot's central axis formed a complex of the second order, such that each straight line is the intersection of two perpendicular tangent planes to the conics used by Minding. The first part of this result was subsequently arrived at by Somoff in 1879.

inclination of the astatic arm to the forces is θ, the arm of the couple is $AB \sin\theta$.

The product of either force into the astatic arm is called the *astatic moment of the couple.* The astatic moment is of course unaltered by any change in position of the body. Representing the astatic moment by K, the actual moment in any position of the body is $K \,.\, \sin\theta$.

The angle θ which the astatic arm makes with the force is called the *astatic angle of the couple.*

Two couples are said to have the *same astatic effect* when they are equivalent in all positions of the body.

For the sake of brevity the couple whose force is P and astatic arm is AB is represented by the symbol (P, AB).

3. *The astatic effect of a couple is not altered if we replace it by another having the same astatic moment, the astatic arms being parallel, and the forces acting in the same directions in space as before.*

Let the astatic arm AB be moved to a new position $A'B'$ in the body. The extremities of the astatic arm of a couple are fixed in the body and move with it, thus as the body is displaced, AB and $A'B'$ continue to be parallel to each other. The astatic angles of the two couples continue therefore to be equal to each other. Since the astatic moments are equal, it follows that the actual moments of the couples are equal. The two couples are therefore equivalent.

It may be noticed that we cannot in general turn the astatic arm of a couple through any angle in the manner explained in Vol. I. Art. 92; for the planes of the couples may not remain parallel to each other, unless the displacements of the body are restricted to be parallel to the original plane of the couples.

4. *To find the astatic resultant of two couples whose forces are parallel but whose astatic arms are inclined at any angle.*

Let AB, $A'B'$ be the astatic arms of the couples, the forces at A, A' being supposed to act in the same direction in space. Through any point O draw OL, OM to represent the directions of AB, $A'B'$ and let the lengths of OL, OM be proportional to the astatic moments of the couples. We shall now prove that the diagonal ON of the parallelogram described on OL, OM will represent in direction the astatic arm of the resultant couple and in length the magnitude of the astatic moment of that couple.

Let the straight lines OL, LN be fixed in the body. By Art. 3

the two couples may be replaced by two others having OL, LN for their astatic arms and having the four forces all equal. The two forces acting at L being equal and opposite may be removed, so that the two given couples are equivalent to two equal and opposite forces acting respectively at O and N. These two forces constitute a single couple having ON for its astatic arm and having its astatic moment proportional to the length of AN. The proposition is therefore proved.

From this proposition we infer that the theorems used to compound forces apply also to compound the astatic arms of couples having their forces parallel. It is hardly necessary to add that the forces of the resultant couple are parallel to those of the two constituents.

5. *To find the astatic resultant of two couples whose astatic arms are parallel but whose forces are inclined at any angle.*

Let AB, $A'B'$ be the parallel astatic arms of the couples, both AB, $A'B'$ pointing in the same direction in the body. Through any point O draw OC parallel to AB and also two straight lines OL, OM parallel to the forces at A and A' and proportional to the astatic moments of the couples. We shall prove that the diagonal ON of the parallelogram OLM represents the moment of the resultant couple, the plane of the couple is parallel to the plane NOC, and the astatic arm is in the direction of OC.

Let the couples be referred to a common astatic arm along OC, the forces at O are then represented by OL and OM. Proceeding as in Art. 4 the results stated are easily seen to be true.

6. **Working rule.** Uniting these two propositions we may construct a rule to resolve or compound couples.

When the forces are parallel we resolve or compound lengths, measured along the astatic arms and proportional to the astatic moments, by the parallelogram law, the new forces being supposed to act parallel to their former directions.

When the arms are parallel we resolve or compound lengths, measured along the directions of the forces and proportional to the astatic moments, by the parallelogram law, the new arms being parallel to their former directions.

7. There is one resolution of a couple which will be found useful afterwards.

Let Ox, Oy, Oz be any set of Cartesian axes not necessarily

rectangular. Let (x, y, z) be the coordinates of any point D, and let $OD = r$. Then a couple whose astatic arm is r and forces $\pm P$ may be resolved into three other couples whose astatic arms are situated in the axes of coordinates and whose lengths are equal to x, y, z. The forces of these couples are parallel to that of the original couple and their astatic moments Px, Py, Pz.

Let us now take any three points A, B, C on the axes and let $OA = a$, $OB = b$, $OC = c$. These three couples may be replaced by three others having OA, OB, OC for their astatic arms. It follows that any force P acting at any point D may be replaced by four parallel forces acting at any four points A, B, C and O whose magnitudes are respectively equal to Px/a, Py/b, Pz/c and $P(1 - x/a - y/b - z/c)$.

Conversely, since these four parallel forces may be compounded into a single force equal to their sum and acting at the centre of gravity, it is evident that they are equivalent to the force P acting at the point (x, y, z). See Vol. I., Art. 80.

8. *Two couples cannot be astatically compounded together into a single resultant couple unless either the four forces are parallel or the two astatic arms are parallel.*

If possible let three couples be in astatic equilibrium. Transfer these parallel to themselves so that one force of each couple acts at the point O. Let OA, OB, OC be the astatic arms, let OP, OQ, OR be the directions of the forces. Then as the body is displaced, OA, OB, OC are fixed in the body, OP, OQ, OR are fixed in space.

If the four forces of any two of the three couples are parallel, the forces of their resultant couple are also parallel to them, by Art. 4. Thus equilibrium could not exist unless all the six forces were parallel to each other. In what follows, we may therefore suppose that no two of the three lines OP, OQ, OR are coincident. In the same way no two of the three OA, OB, OC are coincident.

Place the body so that OC, OR are in one straight line. Since in this position the couples (P, OA), (Q, OB) are in equilibrium, the planes POA, QOB coincide. Thus OA, OB lie in the plane POQ and continue to lie in that plane as the body is turned round OC. It follows that the axis OC must be perpendicular to this plane and therefore to both OA and OB. Similarly OA is perpendicular to both OB and OC.

Supposing as before that OC, OR are in one straight line, it is clear that the body may be turned round OC until OA coincides with OP. The axis OB must then coincide with OQ, for otherwise equilibrium could not exist. Summing up, the axes OA, OB, OC are at right angles and the body can be so placed that the forces of the respective couples act along their astatic axes.

Referring to the figure of Art. 76, Vol. I., we see that if the couple (P, OA) is a stable couple, the couple (Q, OB) must be unstable, for otherwise they would not act in opposite directions when the body is rotated about OC. Similarly by rotating the body about OB we see that (R, OC) is an unstable couple. Therefore (R, OC) cannot balance (Q, OB) when the body is rotated about OA. The three couples cannot therefore be in equilibrium in all positions of the body.

The Central Ellipsoid.

9. *To reduce any number of forces astatically to a single force and three couples.*

Let the forces be P_1, P_2, &c. and let their points of application be M_1, M_2, &c. respectively. Let Ox, Oy, Oz be any axes, not necessarily rectangular, which are fixed in the body and move with it. Let (x, y, z) be the coordinates of the point of application M of any one force P, and let $OM = r$.

Take three arbitrary points A, B, C on the axes of coordinates; let $OA = a$, $OB = b$, $OC = c$. By Art. 7 the force P acting at x, y, z, is equivalent to an equal and parallel force acting at O, together with three astatic couples whose arms are OA, OB, OC respectively, whose astatic moments are Px, Py, Pz and whose forces are parallel to P.

In this way all the forces may be brought to act at the origin parallel to their original directions. These may be compounded together into a single force, whose magnitude and direction in space are the same for all positions of the body. Let us represent this force by R.

Each force P will also give a couple having OA for its astatic arm. Compounding the forces at the extremities of this common arm, all these couples reduce to a single couple. The arm OA of this couple is fixed in the body while the magnitude and direction in space of the forces are the same for all positions of the body. Let us represent the magnitude of either of its forces by F.

The couples having OB, OC, for their astatic arms may be treated in the same way. Their astatic arms also are fixed in the body, while the magnitude and direction in space of the forces are always the same. Let these forces be G and H.

Summing up, we see that a system of forces can be reduced to a principal force R acting at any assumed base point O, together with three couples (F, OA), (G, OB) and (H, OC), having their astatic arms arranged along any three assumed straight lines OA, OB, OC fixed in the body and not all in one plane.

It may be seen that this reasoning, as far as we have gone, is the same as that used in the corresponding proposition when the body is fixed in space (Vol. I., Art. 257). The difference is, that when the body has only one position in space these three couples may be compounded into a single couple. But no single couple

can be found which is equivalent to these, when the body may assume any position in space (Art. 8).

10. Consider any one position of the forces and of the body. In this position let X, Y, Z, be the components along the axes of any force P. To find the resultant force R, we bring all these P's to act at the base O. The force R is therefore the resultant of ΣX, ΣY, ΣZ acting at O along the axes. To avoid the continual recurrence of the symbol Σ it will be convenient to represent these components by X_0, Y_0, Z_0.

To find the force F we seek the resultant of all the forces similar to Px/a acting at A. The force F is therefore the resultant of the three forces $\Sigma Xx/a$, $\Sigma Yx/a$, $\Sigma Zx/a$ acting at A parallel to the axes. In the same way the forces G and H are the resultants of $\Sigma Xy/b$, $\Sigma Yy/b$, $\Sigma Zy/b$ and of $\Sigma Xz/c$, $\Sigma Yz/c$, $\Sigma Zz/c$. It will be found convenient to represent the summations ΣXx, ΣXy &c. by the symbols X_x, X_y, &c.

In this way the three couples (F, a), (G, b), (H, c) are resolved into nine elementary couples whose astatic moments are represented by the constituents of either of the following determinatal figures

$$\begin{aligned}
\text{couple } (F, a) &= \Sigma Xx,\ \Sigma Yx,\ \Sigma Zx = X_x,\ Y_x,\ Z_x\\
\text{couple } (G, b) &= \Sigma Xy,\ \Sigma Yy,\ \Sigma Zy = X_y,\ Y_y,\ Z_y\\
\text{couple } (H, c) &= \Sigma Xz,\ \Sigma Yz,\ \Sigma Zz = X_z,\ Y_z,\ Z_z
\end{aligned}$$

where the common arms of the three couples in the first, second and third rows are OA, OB, OC respectively. Thus the small letter or suffix indicates the axis on which the astatic arm is situated, while the large letter indicates the direction of the force. This convenient notation is the same as that used by Darboux.

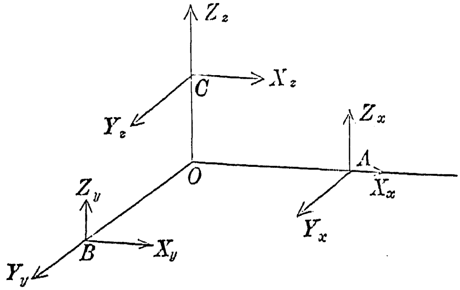
These will be referred to afterwards as the nine elementary couples. Together with X_0, Y_0, Z_0, the three force components, we thus have twelve elementary quantities for each base point.

For the sake of brevity we shall represent the couple (F, a) (G, b), (H, c) by the symbols K_x, K_y, K_z.

As we are chiefly concerned with the astatic moments of the couples, the forces and arms are separately of only slight importance. It is often convenient to *choose the arms of all the couples to be unity and positive.* The signs of the forces alone then determine the signs of the moments. In other cases it is found advantageous to *make the forces of all the couples equal to the force*

R. The forces then divide out of the equations, leaving relations between lengths only.

It will be found useful to remember that the direction ratios of any one of the forces F, G, H are proportional to the constituents of the corresponding row of the determinantal figure. An interpretation of the symbols when taken in columns will be found later on.



The figure represents the relation of the elementary couples to the axes. To avoid complication the forces at O are omitted. The directions of the forces at the extremities A, B, C of the astatic arms are shown by the arrow-head, each arrow-head being marked by the astatic moment of the corresponding couple.

11. **Conditions of equilibrium.** *If a system of forces be in astatic equilibrium each of the twelve elements is zero.*

Resolving parallel to the axes we have $X_0 = 0$, $Y_0 = 0$, $Z_0 = 0$.

Taking moments about the axes of coordinates we have

$$Z_y - Y_z = 0, \quad X_z - Z_x = 0, \quad Y_x - X_y = 0.$$

But the body must be in equilibrium in all positions. Instead of turning the body round any axis, let us turn every force in the opposite direction round a parallel axis through its point of application. First let the rotation be about an axis parallel to x through a right angle. The X forces are all unchanged, but the Y forces now act parallel to z in the positive direction while the Z forces act parallel to y in the negative direction. Hence, writing Y for Z and $-Z$ for Y in the equations of moments already found, we have $$Y_y + Z_z = 0, \quad X_z - Y_x = 0, \quad -Z_x - X_y = 0.$$

Joining these to the preceding equations we find $Z_x = 0$, $X_y = 0$, $X_z = 0$, $Y_x = 0$, i.e. every constituent with an x in it (except X_x) is zero.

In the same way by turning the system round y we find that all the constituents are zero except X_x, Y_y, Z_z. But we also find that $Y_y + Z_z = 0$, $Z_z + X_x = 0$, $X_x + Y_y = 0$. Hence each of the three X_x, Y_y, Z_z is also zero.

Thus all the twelve elements are zero.

12. *If two systems of forces be referred to the same origin and axes they cannot be astatically equivalent unless the twelve elements are equal each to each.*

Let the twelve elements of the two systems be X_x &c., X_x' &c. If we reverse the forces of the second system, the two systems together would be in equilibrium. Hence $X_x - X_x' = 0$, &c. $= 0$.

Thus all the elements are equal each to each.

13. Ex. 1. If the same system of forces can be astatically represented in either of two ways viz. (1) by three forces (F, G, H) acting at (A, B, C) or (2) by three other forces (F', G', H') acting at (A', B', C') prove that (unless the system can be reduced to two astatic forces instead of three) the planes ABC, $A'B'C'$ must coincide.

Let us first suppose that the three forces F, G, H, are not all parallel to one plane. Take the plane $A'B'C'$ as the plane of xy. We have X_z, Y_z, Z_z, the same for both systems. But since the ordinates of the points of application of F', G', H', are zero, each of the three X_z, Y_z, Z_z must be zero. Consider the equation $Z_z = 0$. Place the body in such a position that the forces F, G act parallel to the plane $A'B'C'$. This is possible since a plane can be drawn parallel to any two straight lines. Then by hypothesis the direction of H will not be parallel to the plane $A'B'C'$. The components of the forces F, G, parallel to the axis of z are now zero. Hence Z_z must be zero for the single force H. Thus either $H = 0$ or the ordinate of its point of application is zero. Supposing F, G, H to be all finite, it follows that C lies in the plane $A'B'C'$. By similar arguments we prove that the other points A, B also lie in the same plane $A'B'C'$.

Next, let us suppose all the three forces F, G, H are parallel to one plane. In this case one of the forces as H can be resolved into two components f and g parallel to F and G respectively. Each of the two sets of parallel forces (f, F) and (g, G) can be replaced by a single force at its centre of parallel forces. The system F, G, H can therefore be reduced to two astatic forces.

Ex. 2. If a system of forces F, G, H, acting at the corners of a triangle ABC, can be reduced to two astatic forces F', G' acting at two points A', B', then either the forces F, G, H are all parallel to one plane or the triangle ABC is evanescent.

We need only to examine the case in which F, G, H are all finite, for, if one be zero, the other two are necessarily parallel to one plane.

The system F', G' can be regarded as the limiting case of a triangle of forces F', G', H' acting at the corners of a triangle $A'B'C'$ where H' is zero and the position of C' is arbitrary. If then the forces F, G, H are not all parallel to the same plane it would follow from Ex. 1 that all the corners A, B, C lie in the plane $A'B'C'$. But this is impossible since C' is an arbitrary point unless the triangle ABC is evanescent and lies in the straight line $A'B'$.

14. **The central ellipsoid.** A base point O having been

chosen, the axes Ox, Oy, Oz are arbitrary. We shall now show that there is one system of axes which will enable us to analyse the system of forces more simply than any other.

Let Ox', Oy', Oz' be a second system of axes also fixed in the body. Let A', B', C' be points taken arbitrarily on these axes, let their distances from O be a', b', c'. Let F', G', H' be the forces which act at A', B', C'. We shall suppose both systems of axes to be rectangular.

As the body is moved about, the forces F', G', H' keep their directions in space unaltered, so that as regards the body the points of application and the magnitude of each force are the only elements fixed. Let us then find the magnitude of the force F' which acts at A', the forces at O, A, B, C being regarded as given. To effect this we shall resolve the arms of each of the nine elementary couples along OA', OB', OC', keeping the forces unaltered. We shall reserve for examination only those components whose arms are along OA'.

Let (l, m, n) be the direction cosines of the axis Ox'. Then the groups of couples (X_x, X_y, X_z); (Y_x, Y_y, Y_z); (Z_x, Z_y, Z_z) yield three component couples having their forces parallel to X, Y, Z respectively. Their astatic moments are (Art. 6),

$$X_x l + X_y m + X_z n = L_1,$$
$$Y_x l + Y_y m + Y_z n = L_2,$$
$$Z_x l + Z_y m + Z_z n = L_3.$$

These couples have a common arm OA' and their forces are at right angles. Compounding them we have

$$(F'a')^2 = (X_x l + X_y m + X_z n)^2 + (Y_x l + Y_y m + Y_z n)^2 + (Z_x l + Z_y m + Z_z n)^2.$$

The direction cosines of the force F' are proportional to the three moments L_1, L_2, L_3.

We notice that this expression for $F'a'$ contains only the direction cosines of OA', and does not depend on the position of OB' or OC', except only that these must be at right angles to OA'. We are thus able to consider the couple whose arm is OA' apart from those whose arms are OB' and OC'.

Let us measure along OA' a length OP', such that OP' is inversely proportional to the astatic moment of the couple whose arm is OA'. For convenience we shall suppose the product of OP' and this astatic moment to be unity. Thus $OP' . F'a' = 1$. Let $OP' = \rho$, and let ξ, η, ζ, be the coordinates of P' referred to

the original axes Ox, Oy, Oz. Then $\xi = l\rho$, $\eta = m\rho$, $\zeta = n\rho$. We therefore find for the locus of P' the quadric

$$1 = (X_x\xi + X_y\eta + X_z\zeta)^2 + (Y_x\xi + Y_y\eta + Y_z\zeta)^2 + (Z_x\xi + Z_y\eta + Z_z\zeta)^2.$$

15. This quadric may be regarded as defined by a statical property, viz. if any radius vector be taken as the axis Ox', the astatic moment of the corresponding couple (F', a') is measured by the reciprocal of that radius vector. It follows that whatever coordinate axes Ox, Oy, Oz are chosen we must have the same quadric. The equations to the quadric when referred to different sets of axes may be different, but the quadric itself is always the same. The quadric is therefore to be regarded as fixed in the body. Any point of the body may be chosen as the base O, and every such base has a corresponding quadric whose centre is at the base. This quadric is called *the central ellipsoid of that point.* It is also called *Darboux's ellipsoid.*

16. Let us represent the astatic moment of the couple whose astatic arm is directed from a given base along the radius vector ρ by the symbol K_ρ. In the same way the astatic moments, Fa, Gb and Hc, of the couples whose astatic arms are directed along the axes will be represented by K_x, K_y, K_z. With this notation we have

$$X_x^2 + Y_x^2 + Z_x^2 = F^2a^2 = K_x^2,$$
$$X_y^2 + Y_y^2 + Z_y^2 = G^2b^2 = K_y^2,$$
$$X_z^2 + Y_z^2 + Z_z^2 = H^2c^2 = K_z^2;$$
$$X_yX_z + Y_yY_z + Z_yZ_z = K_yK_z \cos\alpha,$$
$$X_zX_x + Y_zY_x + Z_zZ_x = K_zK_x \cos\beta,$$
$$X_xX_y + Y_xY_y + Z_xZ_y = K_xK_y \cos\gamma;$$

where α, β, γ, are the angles between the directions of the forces (G, H), (H, F), (F, G) of the couples K_x, K_y, K_z.

Expanding the squares, the equation to the central ellipsoid of the origin may be written in the form

$$K_x^2\xi^2 + K_y^2\eta^2 + K_z^2\zeta^2 + 2K_yK_z\cos\alpha\,\eta\zeta + 2K_zK_x\cos\beta\,\zeta\xi + 2K_xK_y\cos\gamma\,\xi\eta = 1.$$

Also if K' be the moment of the couple corresponding to the arm OA', whose direction cosines are l, m, n, we have

$$K'^2 = K_x^2l^2 + K_y^2m^2 + K_z^2n^2 + 2K_yK_zmn\cos\alpha + 2K_zK_xnl\cos\beta + 2K_xK_ylm\cos\gamma.$$

It may be useful to state the rule by which the sign of any of the astatic moments K_x, K_y, K_z is determined. The directions of the forces being fixed in space, there is for each line of action a positive and a negative direction determined by reference to some axes fixed in space. The astatic arms are measured in the body, and for each of these also there is a positive and a negative direction. Now

imagine the couple moved parallel to itself until either extremity of its astatic arm is placed at the origin, so that one force acts at the origin. The moment is then the product of the astatic arm into the other force, when each is taken with its proper sign.

17. Show that the discriminant of the central ellipsoid at the origin is equal to $(6VFGH)^2$, where V is the volume of the tetrahedron $OABC$.

Prove also that the minors of the coefficients ξ^2, η^2, ζ^2 in the discriminant are $(K_yK_z \sin\alpha)^2$, $(K_zK_x \sin\beta)^2$ and $(K_xK_y \sin\gamma)^2$, respectively.

If parallels to the directions of the forces F, G, H are drawn from the centre of a sphere to cut the surface, the arcs joining the points of intersection form a spherical triangle whose sides are α, β, γ. If θ, ϕ, ψ be the opposite angles, the minors of the coefficients of $\eta\zeta$, $\zeta\xi$, $\xi\eta$ in the discriminant are respectively

$-K_yK_zK_x^2 \sin\beta \sin\gamma \cos\theta$, $-K_zK_xK_y^2 \sin\gamma \sin\alpha \cos\phi$ and $-K_xK_yK_z^2 \sin\alpha \sin\beta \cos\psi$.

Ex. 2. An astatic arm OP moves about any given base point O so that its corresponding astatic moment is constant. Show that OP traces out a cone in the body coaxial with the central ellipsoid at O.

Ex. 3. If Ox, Oy, Oz be any rectangular axes meeting at a fixed origin O, K_x, K_y, K_z the corresponding astatic moments, prove that $K_x^2 + K_y^2 + K_z^2$ is invariable for all such axes.

Since this expression is the first invariant of the central ellipsoid at O the property follows at once. It also follows from the geometrical property of an ellipsoid, that the sum of the squares of the reciprocals of three diameters at right angles is constant.

18. If we refer the central ellipsoid to its principal diameters as axes of reference, the equation loses the terms containing the products of the coordinates. If F, G, H represent the forces of the three couples for this position of the axes, the equation is

$$F^2a^2\xi^2 + G^2b^2\eta^2 + H^2c^2\zeta^2 = 1.$$

The quadric is therefore in general an ellipsoid. If one of the three forces is zero, i.e. if one of the couples is absent, the quadric reduces to a cylinder.

Since the terms containing the products $\xi\eta$, $\eta\zeta$, $\zeta\xi$ are absent, it follows that if the three forces F, G, H are all finite, their directions are at right angles to each other. If one force is zero, the other two must be at right angles.

Summing up, we see that *whatever point of the body we choose as base, there are always three straight lines at right angles, fixed in the body, such that, when these are taken as the astatic arms of the couples, the forces of the couples act in directions at right angles to each other and are fixed in space.*

In this way we have for each base point two convenient systems of rectangular axes, one fixed in the body, viz. the astatic arms of the couples, the other fixed in space, viz. the directions of the forces.

The axes fixed in the body are called the *principal axes of the base*. The couples are then called the *principal couples*.

19. **The initial position.** The base point O being regarded as fixed, and the body referred to principal axes, it is evident that we may turn the body about O until the system of axes fixed in the body coincides in position with the system fixed in space.

The peculiarity of this position of the body is that the forces of each of the three couples act along the astatic arm of that couple. The moments of the couples are therefore zero. The forces P_1, P_2, &c. of the given system reduce to the single resultant R whose line of action passes through the given base.

This is called an *initial position* of the body and the couples are then said to be in their *zero positions*.

The body being placed in an initial position, it is clear that if we turn it round any one of the astatic arms through two right angles, the same property will recur again, i.e. the force of each couple will act along its astatic arm. Thus any base being given there are at least four corresponding initial positions.

Though in all these four positions of the body the two systems of axes coincide in position, yet the positive direction of an axis of one system may be the same as either the positive or the negative direction of an axis of the other system. It is usual to choose the positive directions of one system so that in one of these four positions of the body the two systems of axes may have the same positive directions as well as coincide in position. This initial position is called the *positive initial position*.

20. When the body is placed in a positive initial position the nine elementary couples described in Art. 10 are reduced to

$$\begin{array}{lll} X_x & Y_x = 0 & Z_x = 0, \\ X_y = 0 & Y_y & Y_z = 0, \\ X_z = 0 & Y_y = 0 & Z_z. \end{array}$$

The equation to the central ellipsoid then takes the simple form
$$X_x^2\xi^2 + Y_y^2\eta^2 + Z_z^2\zeta^2 = 1.$$

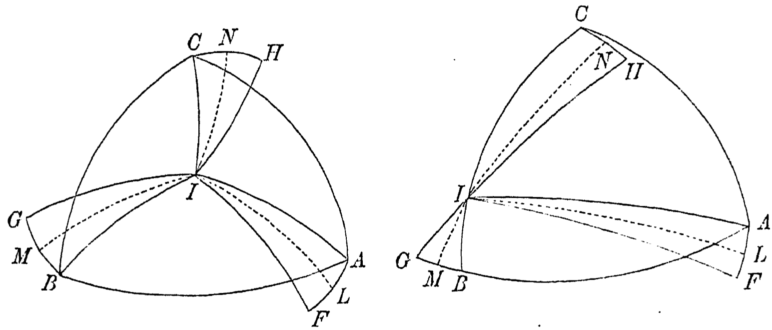
If (l, m, n) be the direction cosines of any other arm OA' the direction cosines of the force F' acting at its extremity are proportional to
$$X_x l, \quad Y_y m, \quad Z_z n,$$
and
$$(F'a')^2 = X_x^2 l^2 + Y_y^2 m^2 + Z_z^2 n^2.$$

Thus the direction and magnitude of F' have been found. If

the body is now moved into any other position, F' continues to act in the same direction in space and therefore continues to make the same angles with F, G, H that it made in the initial position.

21. *There are no other positions besides the four initial positions in which a body can be placed so that the system of forces may reduce to a single resultant which passes through the given base, except when the central ellipsoid at the given base point is a surface of revolution.*

Let OA, OB, OC be the principal axes at the given base O. Let OF, OG, OH be three straight lines at right angles drawn parallel to the forces of the corresponding couples. In order to use conveniently the formulæ of spherical trigonometry we suppose these axes to cut the surface of a sphere whose centre is at O in the six points A, B, C, F, G, H. The planes of the couples are the planes which contain the astatic arms and the forces, and are therefore the planes of the spherical arcs AF, BG, CH. If their astatic moments are $K_x=Fa$, $K_y=Gb$, $K_z=Hc$ their moments in any position of the body are $K_x \sin AF$, $K_y \sin BG$ and $K_z \sin CH$.



When the body is in an initial position the spherical triangles coincide. Starting from this position, the body may be brought into any other by turning it round some axis OI. If this axis intersect the sphere in I, the spherical arcs IA, IB, IC are respectively equal to IF, IG, IH, and if 2ω is the angle of rotation, the angles AIF, BIG, CIH are each equal to 2ω. Join AF, BG, CH by arcs of great circles and draw the perpendicular arcs IL, IM, IN.

If this position of the body can be one of equilibrium when the base is fixed, the three couples must balance each other. Resolving the axis of each of these along and perpendicular to OI, the moments of the three latter components are respectively $K_x \sin AF \cos IL$, $K_y \sin BG \cos IM$, and $K_z \sin CH \cos IN$. Since the three components are in equilibrium, these moments must be proportional to $\sin MIN$, $\sin NIL$, $\sin LIM$ that is to $\sin BIC$, $\sin CIA$ and $\sin AIB$.

For brevity let α, β, γ represent the arcs IA, IB, IC. Since BC is a right angle we have
$$\cos\beta\cos\gamma+\sin\beta\sin\gamma\cos BIC=\cos BC=0$$
$$\begin{aligned}\therefore\ \sin^2\beta\sin^2\gamma\ \sin^2 BIC&=\sin^2\beta\sin^2\gamma-\cos^2\beta\cos^2\gamma\\&=1-\cos^2\beta-\cos^2\gamma\\&=\cos^2\alpha.\end{aligned}$$
Again,
$$\sin AF\cos IL=2\sin\tfrac{1}{2}AF\cos\alpha=2\sin\alpha\cos\alpha\sin\omega.$$

Similar expressions hold for the other angles.

Substituting these values in the condition of equilibrium, and dividing out the

common factors, we have $K_x^2 = K_y^2 = K_z^2$. Thus the proposed position of the body cannot be one of equilibrium when the base is fixed unless the ellipsoid is a sphere.

This argument assumes that none of the factors divided out are zero. We must therefore examine separately the case in which I lies on one of the principal planes. If I lies on BC, the first component is zero, and the other two are $K_y \sin BG \cos IM$ and $K_z \sin CH \cos IN$. The condition of equilibrium is that these moments should be equal; hence $K_y^2 \sin^2\beta \cos^2\beta = K_z^2 \sin^2\gamma \cos^2\gamma$.
Since β and γ are complementary, this requires that $K_y^2 = K_z^2$, i.e. the ellipsoid is one of revolution.

Lastly, if I is at the point C, each of the three component couples is zero. The component having OI for its axis is then the sum or difference of the couples $K_x \sin 2\omega$, $K_y \sin 2\omega$. Since this component also must vanish we again have $K_x^2 = K_y^2$, i.e. the ellipsoid is one of revolution.

22. Ex. 1. The body being placed in a positive initial position, prove that the direction of F' is parallel to the normal to the ellipsoid $X_x\xi^2 + Y_y\eta^2 + Z_z\zeta^2 = 1$ drawn at the point where OA' cuts the ellipsoid. This ellipsoid is called the second central ellipsoid of Darboux.

Ex. 2. The body being placed in a positive initial position, a straight line OQ is drawn from the base parallel and proportional to the force F' for all positions of OA' in the body. Prove that the locus of Q is the ellipsoid

$$\left(\frac{\xi}{X_x}\right)^2 + \left(\frac{\eta}{Y_y}\right)^2 + \left(\frac{\zeta}{Z_z}\right)^2 = 1.$$

This is called the third central ellipsoid of Darboux.

Prove also that, if the arms OA', OB', OC' be at right angles, the corresponding forces F', G', H' are parallel to a system of conjugate diameters in this third ellipsoid. This and the last example are due to Darboux.

Ex. 3. When the body is in a positive initial position for any base, prove that the direction of the force corresponding to any astatic arm OA' is parallel to the eccentric line of OA' in the central ellipsoid of the given base.

The Central Plane and the Central Point.

23. *To compare the central ellipsoids at different points of the body.*

Suppose the forces to be referred to any base O and any axes Ox, Oy, Oz, and that the nine elementary couples and the three force-components are known for these axes. We shall now find the corresponding quantities when some point O', whose coordinates are (p, q, r), is taken as the base.

Through O' we draw axes $O'x'$, $O'y'$, $O'z'$ parallel to (x, y, z). The nine elementary couples may be transferred to these new axes without any change (Art. 3). But the three force-components will introduce new couples. By Art. 7 the component X_0 acting at O may be transferred to the origin O' if we introduce the new couples $(X_0, -p)$, $(X_0, -q)$, $(X_0, -r)$, the coordinates of O referred to O' being $(-p, -q, -r)$. Similar reasoning applies to the components Y_0, Z_0. Hence we have for the nine elementary couples at O'

$$X_x' = X_x - X_0 p, \qquad Y_x' = Y_x - Y_0 p, \qquad Z_x' = Z_x - Z_0 p,$$
$$X_y' = X_y - X_0 q, \qquad Y_y' = Y_y - Y_0 q, \qquad Z_y' = Z_y - Z_0 q,$$
$$X_z' = X_z - X_0 r, \qquad Y_z' = Y_z - Y_0 r, \qquad Z_z' = Z_z - Z_0 r.$$

The equation to the central ellipsoid at O' is therefore, by Art. 14,

$$\begin{aligned} &\{(X_x - X_0 p)\,\xi' + (X_y - X_0 q)\,\eta' + (X_z - X_0 r)\,\zeta'\}^2 \\ &+ \{(Y_x - Y_0 p)\,\xi' + (Y_y - Y_0 q)\,\eta' + (Y_z - Y_0 r)\,\zeta'\}^2 \\ &+ \{(Z_x - Z_0 p)\,\xi' + (Z_y - Z_0 q)\,\eta' + (Z_z - Z_0 r)\,\zeta'\} = 1\,; \end{aligned}$$

the origin of the running coordinates ξ', η', ζ' being O'.

24. If the principal force R is zero, we have $X_0 = 0$, $Y_0 = 0$, $Z_0 = 0$. In this case the central ellipsoid at O' is the same as that at O. Thus the central ellipsoids at all base points are similar and similarly situated.

25. **The Central Plane.** If the principal force R be not zero the form of the central ellipsoid will depend on the position of the base point. We notice that the three planes

$$\begin{aligned} (X_x - X_0 p)\,\xi' + (X_y - X_0 q)\,\eta' + (X_z - X_0 r)\,\zeta' &= 0, \\ (Y_x - Y_0 p)\,\xi' + (Y_y - Y_0 q)\,\eta' + (Y_z - Y_0 r)\,\zeta' &= 0, \\ (Z_x - Z_0 p)\,\xi' + (Z_y - Z_0 q)\,\eta' + (Z_z - Z_0 r)\,\zeta' &= 0 \end{aligned}$$

are conjugate planes.

If the central ellipsoid is a cylinder all the conjugate planes pass through the axis of the cylinder, and the equations to the three conjugate planes are then not independent. We thus have the determinantal equation

$$\begin{vmatrix} X_x - X_0 p, & X_y - X_0 q, & X_z - X_0 r, \\ Y_x - Y_0 p, & Y_y - Y_0 q, & Y_z - Y_0 r, \\ Z_x - Z_0 p, & Z_y - Z_0 q, & Z_z - Z_0 r, \end{vmatrix} = 0 \quad \ldots(1).$$

This equation may be written in the form

$$\begin{vmatrix} X_0 & X_x & X_y & X_z \\ Y_0 & Y_x & Y_y & Y_z \\ Z_0 & Z_x & Z_y & Z_z \\ 1 & p & q & r \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

When p, q, r are regarded as the running coordinates, this is evidently the equation to a plane. *The peculiarity of this plane is that, if any point on it is chosen as base, the central ellipsoid is a cylinder.* This plane is called *the central plane.*

26. Since the central ellipsoid at every point is fixed in the body the locus of base points at which the ellipsoid is a cylinder is also fixed. *The central plane is therefore fixed in the body.* In discussing its properties we may put the body into any position we please.

Take any point O on the central plane as base, and let the body be placed in an initial position. By Art. 20 all the nine elementary couples, except X_x, Y_y, Z_z, are zero. Since the ellipsoid is a cylinder one of the three X_x, Y_y, Z_z is also zero, say $X_x = 0$. Substituting in the second form of the equation to the central plane given in Art. 25, we see that it becomes $pX_0 Y_y Z_z = 0$. If any one of the three X_0, Y_y, Z_z is zero, the equation to the plane is indeterminate, but if all these are finite, the equation to the central plane is $p = 0$. It follows therefore that *the infinite axis of the central ellipsoid at any point of the central plane is perpendicular to that plane.*

27. This leads to a simplified reduction of the forces P_1, P_2, &c. Let us take the base of reference O at any point of the central plane, and the principal diameters of the central cylinder as axes of coordinates. The moment of that principal couple whose astatic axis is along the infinite axis of the cylinder is measured by the reciprocal of that axis, and is therefore zero. Thus all the forces have been reduced to two couples (instead of three) and a force R. The astatic arms of the couples lie in the central plane and the forces of one couple are perpendicular to those of the other.

28. **The Central Point.** It has been proved in Art. 10 that a system of forces may be reduced to a principal force R at the base of reference and three couples having their arms directed along any three straight lines at right angles. Let us now enquire if a base O' can be found such that each of the forces of the couples are perpendicular to the principal force.

If one system of axes $O'A$, $O'B$, $O'C$ at any base O' possess this property, then every system of axes at that base will also possess the same property. To prove this, let $O'A'$, $O'B'$, $O'C'$ be any other such system of axes. To deduce the forces at A', B', C' from those at A, B, C, we resolve the arms OA, OB, OC in the directions OA', OB', OC' and transfer the forces parallel to themselves, see Art. 6. Since each of the forces at A, B, C is

perpendicular to the force R, it follows that the forces at A', B', C', which are compounded of these, are also perpendicular to R.

Let Ox, Oy, Oz be any given rectangular axes, and let p, q, r be the coordinates of O'. Through O' draw a system of axes $O'x'$, $O'y'$, $O'z'$ parallel to Ox, Oy, Oz. Then, by what has just been proved, the couples corresponding to these axes must have their forces perpendicular to R. If the nine corresponding elementary couples are X_x' &c., the conditions of perpendicularity are

$$X_0X_x' + Y_0Y_x' + Z_0Z_x' = 0,$$

and two similar equations obtained by writing y and z for x in the suffixes. Substituting for X'_x, &c. their values given in Art. 23, we find

$$R^2p = X_0X_x + Y_0Y_x + Z_0Z_x,$$
$$R^2q = X_0X_y + Y_0Y_y + Z_0Z_y,$$
$$R^2r = X_0X_z + Y_0Y_z + Z_0Z_z.$$

Since these give only one set of values for p, q, r there is but one point which possesses the given property. This point is called *the central point.*

29. *The central point lies on the central plane.* To prove this let us consider the principal axes at the central point. Since the forces of the three couples are at right angles to each other, they cannot all, if finite, be perpendicular to the principal force. One of these must therefore vanish. The central ellipsoid is therefore a cylinder, i.e. the central point lies on the central plane.

That the central point lies in the central plane may also be proved by substituting its coordinates in the equation (2) of the central plane found in Art. (25). These coordinates p, q, r are given in Art. 28, and a simple inspection shows that the equation is satisfied.

Thus it appears that *there is a certain point, lying on the central plane, such that the forces of the two principal couples at that point are at right angles to each other and to the principal force. This point is called the central point.*

The central point in the three dimensional theory has not the same signification as the central point defined in Vol. I., Art. 160, with reference to two dimensions. In the latter the displacements of the body are confined to one plane, and for such displacements the single resultant always passes through a central point fixed in the body. In the former the displacements are unrestricted so that the lines of action of the forces do not necessarily remain in one plane.

The preceding theorems on the central plane and central point are generally given in treatises on Astatics, though the demonstrations in each may be different.

30. *We may express the formulæ for the coordinates of the central Point in the form of a working rule.*

As already explained in Art. 9 the forces are represented by P_1, P_2, &c. Their points of application are M_1, M_2, &c. and their coordinates are (x_1, y_1, z_1), (x_2, y_2, z_2), &c. Also let the direction cosines of P_1, P_2, &c. be respectively (a_1, b_1, c_1), (a_2, b_2, c_2), &c.

Then
$$X_x = P_1a_1x_1 + P_2a_2x_2 + \dots \qquad X_0 = P_1a_1 + P_2a_2 + \dots$$
$$Y_x = P_1b_1x_1 + P_2b_2x_2 + \dots \qquad Y_0 = P_1b_1 + P_2b_2 + \dots$$
$$Z_x = P_1c_1x_1 + P_2c_2x_2 + \dots \qquad Z_0 = P_1c_1 + P_2c_2 + \dots$$

Let θ_{12}, θ_{13}, &c., be the inclinations of the forces (P_1, P_2) (P_1, P_3) &c. Then
$$\cos\theta_{12} = a_1a_2 + b_1b_2 + c_1c_2 \text{ &c.}$$

Substituting in the expression for p, Art. 28, we have
$$p = \frac{P_1Q_1x_1 + P_2Q_2x_2 + \dots}{P_1Q_1 + P_2Q_2 + \dots}$$

where
$$Q_1 = P_1 + P_2\cos\theta_{12} + P_3\cos\theta_{13} + \dots$$
$$Q_2 = P_1\cos\theta_{12} + P_2 + P_3\cos\theta_{23} + \dots$$
&c. &c.

It is evident that Q_1 is the sum of the resolved parts of all the forces in the direction P_1, Q_2 is the sum of the resolved parts in the direction P_2, and so on.

The equation just arrived at is the common formula for the centre of gravity of weights P_1Q_1, P_2Q_2 &c. Similar equations hold for q and r. Hence we have this rule. *To find the central point of any number of forces, we first multiply each force by the sum of the resolved parts of all the forces along the direction of that force. We then place weights proportional to these products at the points of application of the forces. The centre of gravity of these weights is the central point required.*

31. Ex. Show that the equation to the central plane, referred to any axes, when expressed in terms of the forces and their mutual inclinations takes the form
$$L_x\xi + L_y\eta + L_z\zeta = M$$
where
$$M = \Sigma P_1P_2P_3V_{123}\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix} \text{ and } V_{123} = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}.$$

The coefficient L_x is derived from M by writing unity for each of the x's in the determinant, L_y is derived from M by writing unity for each y, and so on.

To prove this, we start with the equation (2) of the central plane given in Art. 25 and make the same substitutions as in Art. 30. On writing down the determinant it will be seen that the determinants L_x, L_y, L_z may be obtained from the determinant M by the rule just stated. The determinant M when expanded takes

the form of a series of products of triplets of the forces. To find the coefficient of $P_1P_2P_3$ we put all the other forces equal to zero; the determinant then assumes the known form of the product of the two determinants just written down.

32. **Summary.** It will be convenient if we now sum up shortly the gradual steps made in reducing a system of forces to its simplest equivalents.

1. In Art. 9 the forces were reduced to a force R at an arbitrary base point O together with three couples whose arms Ox, Oy, Oz are arbitrary.

2. In Art. 18 it was shown that at the arbitrary base the arms Ox, Oy, Oz could be chosen at right angles to each other so that the forces of each couple are at right angles to the forces of the other two couples. These arms are called the principal axes at O and are fixed in the body.

3. In Art. 25 it was shown that, if the base point O is placed anywhere on a certain plane fixed in the body, the forces can be reduced to the single force R together with *two* couples. The arms of these couples are at right angles and lie in the plane. The forces also of each couple are perpendicular to those of the other. This plane is called the central plane.

4. In Art. 28 it was shown that if the base point is placed at a certain point on the central plane the forces of the couples are perpendicular to the force R. Thus the forces of the original system can finally be reduced to a force R together with two couples whose arms are at right angles and such that the forces of each couple are not only perpendicular to those of the other but are also perpendicular to the force R. This base point is called the central point.

The principal axes at the central point are two straight lines lying in the central plane and a third perpendicular to that plane. The two former are called *the central lines* of the central plane. The latter is sometimes called the central axis. But it must not be confused with Poinsot's central axis with which it coincides only when the body is properly placed. It bears indeed a certain resemblance to Poinsot's central axis, for the system is reduced to a force and two couples (instead of one) such that the forces of the couples are perpendicular to the force.

33. **Analogy to Moments of inertia.** Ex. 1. If K be the astatic moment of the couple corresponding to any astatic arm OP drawn from the central point O, prove that the astatic moment K' of the couple corresponding to any parallel arm $O'P'$ drawn from any point O' is given by $K'^2=K^2+R^2p^2$ where p is the projection of OO' on either astatic arm.

Thus, a motion of the base O in a direction perpendicular to the astatic arm does not alter the magnitude of the astatic moment but a motion along the arm from the central point increases the moment.

Ex. 2. If K_1, K_2, K_3 be the astatic moments corresponding to the principal astatic axes Ox, Oy, Oz drawn from any point O, prove that the astatic moment K corresponding to any arm OP making angles α, β, γ with the axes is given by

$$K^2=K_1^2\cos^2\alpha+K_2^2\cos^2\beta+K_3^2\cos^2\gamma.$$

It appears from these two propositions that the theory of astatic moments of couples has an analogy to the theory of moments of inertia. The square of the astatic moment about an arm drawn from O in any direction OP corresponds to the moment of inertia of a rigid body with regard to a *plane* drawn through O perpendicular to OP. By noticing this correspondence we may deduce the analogous propositions in the two theories one from the other.

It is clear from these two propositions that the mass of the rigid body is analogous to the square of the principal force R, and that the centre of gravity

must be at the central point. For any base in the central plane the moment of the couple whose astatic arm is perpendicular to that plane is zero, hence the rigid body must be a lamina whose plane is the central plane of the forces. See a note at the end of the volume.

As moments of inertia are usually studied in close connection with Rigid Dynamics it is premature to use this analogy as a means of proof in a treatise on Statics.

The Confocals.

34. *To investigate the mode in which the central ellipsoids at different bases are arranged about the central point.*

Let the central point be chosen as the origin and the principal diameters of the central ellipsoid as axes of coordinates. Let the infinite axis be the axis of x, then the plane of yz is the central plane.

As we are enquiring into the positions of the neighbouring central ellipsoids, and as these are fixtures in the body, we may put the body itself into any position we may find convenient. Let it be placed in its positive initial position with the central point as the base.

In this position all the nine elementary couples are zero, except Y_y and Z_z. Also $X_0 = R$, $Y_0 = 0$, $Z_0 = 0$. The central ellipsoid at the origin is $Y_y^2\eta^2 + Z_z^2\zeta^2 = 1$..................(1).

The central ellipsoid at any point O' whose coordinates are p, q, r, is

$$Y_y^2\eta'^2 + Z_z^2\zeta'^2 + R^2(p\xi' + q\eta' + r\zeta')^2 = 1, \dots\dots\dots (2),$$

where (ξ', η', ζ') are referred to axes meeting at O' parallel to the axes x, y, z, Art. 23.

Let an astatic arm $O'A'$ move about O' so that the corresponding couple (F', OA') has a constant astatic moment equal to M, and in any position let (l, m, n) be its direction cosines. Then, since the moment M (Art. 14) is the reciprocal of the corresponding radius vector of the central ellipsoid, we see that l, m, n are connected together by the relation

$$Y_y^2 m^2 + Z_z n^2 + R^2(pl + qm + rn)^2 = M^2$$

$$\therefore M^2l^2 + (M^2 - Y_y^2)m^2 + (M^2 - Z_z^2)n^2 = R^2(pl + qm + rn)^2 \dots (3).$$

Now, after division by R^2, the left-hand side of equation (3) expresses the square of the perpendicular drawn from the central point on a tangent plane to the ellipsoid

$$\frac{\xi^2}{M^2} + \frac{\eta^2}{M^2 - Y_y^2} + \frac{\zeta^2}{M^2 - Z_z^2} = \frac{1}{R^2}; \dots\dots\dots\dots\dots (4)$$

and the right-hand side of (3) expresses the square of the perpendicular from the central point on a plane through O' parallel to that tangent plane. The equation (3) therefore shows that this tangent plane passes through O'. Hence we infer *that if $O'A'$ move about O', so that the corresponding astatic moment is constant and equal to M, then $O'A'$ is always perpendicular to a tangent plane drawn from O' to touch the confocal* (4).

These tangent planes all touch the enveloping cone of the confocal (4), and the axis $O'A'$ traces out the reciprocal cone of this enveloping cone. These two cones are known to be co-axial and their axes (Art. 17, Ex. 2) are in the same directions as those of the central ellipsoid at O'.

If M is so chosen that the confocal (4) passes through the point O', the enveloping cone becomes the tangent plane and therefore the cone traced out by $O'A'$ reduces to the normal at O'.

Hence *the principal diameters of the central ellipsoid at any point O' are the three normals to the three quadrics which pass through O' confocal to the quadric* (4). *Also the astatic moments of the three corresponding couples are the values of M given by the cubic* (4) *when we write for ξ, η, ζ the coordinates of O'.*

35. *Instead of using the three confocals we may use any one of them, say the ellipsoid.* By known properties of solid geometry the three normals at any point O' are (1) the normal to the ellipsoid, (2) parallels to the principal diameters of the section of the ellipsoid diametral to OO'.

Let M_1, M_2, M_3 be the three values of M given by the cubic (4), M_1 being the greatest. Let D_2, D_3 be the lengths of the principal semidiameters of the section of the ellipsoid, D_2 being parallel to the normal at O' to the confocal M_2, and D_3 parallel to the normal to M_3. Then it is known by solid geometry that

$$D_2^2 = M_1^2 - M_2^2,$$
$$D_3^2 = M_1^2 - M_3^2.$$

Thus M_2, M_3 are known in terms of M_1 and quantities connected with the ellipsoid.

36. As these confocals play an important point in the theory of astatic forces, it is necessary to state distinctly their position.

Let the body be referred to the central point as origin, and the principal diameters of the central cylinder as axes, the plane of yz

being the central plane. Let K_2, K_3 be the astatic moments of the couples whose astatic arms are along y and z. These astatic moments are the same for all positions of the body and are represented by Y_y and Z_z when the body is in its initial position. The equation to the confocals is therefore

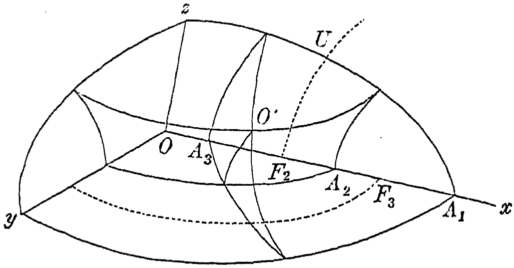
$$\frac{\xi^2}{M^2}+\frac{\eta^2}{M^2-K_2^2}+\frac{\zeta^2}{M^2-K_3^2}=\frac{1}{R^2}.$$

The focal conics of these are obtained in the usual manner by putting $M=K_2$, $\eta=0$; $M=K_3$, $\zeta=0$; and $M=0$, $\xi=0$. We thus have

$$\frac{\xi^2}{K_2^2}-\frac{\zeta^2}{K_3^2-K_2^2}=\frac{1}{R^2}, \qquad \eta=0;$$

$$\frac{\xi^2}{K_3^2}+\frac{\eta^2}{K_3^2-K_2^2}=\frac{1}{R^2}, \qquad \zeta=0;$$

$$-\frac{\eta^2}{K_2^2}-\frac{\zeta^2}{K_3^2}=\frac{1}{R^2} \qquad \xi=0.$$



If we take as the standard case $K_3 > K_2$, the first is a hyperbola, the second an ellipse, and the third is imaginary. The two first are represented in the diagram by the dotted lines. These conics will be referred to as the focal conics, and a straight line intersecting both conics may be called a *focal line*.

The figure represents the positive octant of a set of confocal quadrics intersecting in O'. The semi x-axes are represented by OA_1, OA_2, OA_3 and are respectively equal to M_1/R, M_2/R, M_3/R. As is well known the vertices F_3, F_2 of the two focal conics lie between A_1, A_2 and A_3. We have $OF_2=K_2/R$, $OF_3=K_3/R$.

If $K_2=0$, the ellipsoid and the hyperboloid of one sheet are surfaces of revolution. The hyperboloid of two sheets reduces to any two planes through Oz, and the hyperbolic conic becomes the axis of z. The central plane is now indeterminate and is any plane through the astatic arm of K_3.

If both $K_2=0$ and $K_3=0$, the ellipsoid becomes a sphere, one hyperboloid is a right cone, and the other any two planes through the axis of the cone.

37. **Theorem on focal lines.** *A straight line is drawn from any point P on one focal conic to any point Q on the other, it is required to prove that*

$$R^2\rho^2 = K_2^2 a_2^2 + K_3^2 a_3^2,$$

where a_1, a_2, a_3 are the direction cosines of PQ, and ρ is the perpendicular distance from the origin.

We know that the tangent planes drawn through any right line to the two confocals which that line touches are at right angles to each other, see *Salmon's Solid Geometry*, Art. 172. Since the focal conics are evanescent confocals, the planes through PQ and the tangents at P and Q to the conics are at right angles. If p, p' are the perpendiculars on these planes, l, m, n; l', m', n' their direction cosines we have

$$R^2p^2 = K_2^2l^2 - (K_3^2 - K_2^2)\, n^2, \qquad R^2p'^2 = K_3^2l'^2 + (K_3^2 - K_2^2)\, m'^2.$$

$$\therefore\ R^2\rho^2 = R^2(p^2 + p'^2) = K_2^2(l^2 + n^2 - m'^2) + K_3^2(l'^2 + m'^2 - n^2).$$

Since the straight lines p, p' and PQ are mutually at right angles, this becomes

$$K_2^2(1 - m^2 - m'^2) + K_3^2(1 - n^2 - n'^2) = K_2^2a_2^2 + K_3^2a_3^2.$$

The theorem may be more easily proved by taking as the coordinates of P and Q (x, y, z) and (x', y', z') where

$$Rx = K_2 \sec\theta, \qquad Ry = 0, \qquad Rz = (K_3^2 - K_2^2)^{\frac{1}{2}} \tan\theta,$$

$$Rx' = K_3 \cos\phi, \qquad Ry' = (K_3^2 - K_2^2)^{\frac{1}{2}} \sin\phi, \qquad Rz' = 0.$$

The direction cosines a_2, a_3 and the length ρ may then be found by elementary formulæ, and it will be seen that the relation to be proved is satisfied.

It follows from this theorem that every focal line is a generator of the right circular cylinder whose radius is ρ and whose axis passes through the common centre of the conics and is parallel to the focal line.

Ex. 1. Show that four real focal lines can be drawn parallel to a given straight line.

Let a generator parallel to the given straight line travel round the hyperbolic conic and trace out a cylinder. This will cut the plane of the other conic in a hyperbola. Each branch of this hyperbola passes inside the elliptic conic, because it goes through the focus; it therefore cuts the ellipse in two points.

Ex. 2. If a straight line PQ intersect one focal conic and if its distance from the central point be ρ, where ρ is given in the theorem above, show that that straight line will intersect the other conic also.

If possible let PQ intersect one focal conic in P and not intersect the other. Describe two cylinders whose bases are the focal conics and whose generators are parallel to PQ. By Ex. 1 these intersect in four lines, and each of these four is also a generator of the right circular cylinder whose radius is ρ. Now by supposition PQ, lies on one of the elliptic cylinders and also on the circular cylinder, hence these two quadric cylinders intersect each other in five lines, which is impossible.

Ex. 3. The locus of all the straight lines drawn from any given point P on the hyperbolic conic to intersect the elliptic conic is a right cone, the tangent of whose semi-angle is $(K_3^2 - K_2^2)/K_3Rz$ where z is the ordinate of P.

Ex. 4. Show that four real focal lines can be drawn through a given point P, and that they are the intersections of the two quadric cones

$$\frac{(p\zeta - r\xi)}{K_3^2} + \frac{(q\zeta - r\eta)^2}{K_3^2 - K_2^2} = \frac{\zeta^2}{R^2}$$

$$\frac{(p\eta - q\xi)^2}{K_2^2} - \frac{(q\zeta - r\eta)^2}{K_3^2 - K_2^2} = \frac{\eta^2}{R^2}$$

where (p, q, r) are the coordinates of P and ξ, η, ζ are referred to parallel axes meeting at P.

Ex. 5. Prove that the circular sections of the central ellipsoid whose centre is at O' are perpendicular to the generating lines at O' of the hyperboloid of one sheet. [Darboux.]

Ex. 6. If the base is situated on one of the principal planes at the central point, show that one principal axis at that base is perpendicular to that plane and the astatic moment of the corresponding couple is the same for all base points in that plane.

Ex. 7. If the base is situated on one of the principal axes at the central point, prove that the three principal axes at the base are parallel to those at the central point.

Ex. 8. If a straight line is a principal axis at every point of its length prove that it is one of the principal axes at the central point.

Ex. 9. Find the locus of the base point O' at which the central ellipsoid is a surface of revolution.

In order that two of the three quantities M_1, M_2, M_3, in Art. 35 may be equal we must have either $D_2=0$ or $D_2=D_3$. In the first case O' lies on the elliptic focal conic. In the second case O' is at an umbilicus U and the locus is therefore the hyperbolic focal conic. In both cases the unequal axis is a tangent to the focal conic.

The same results follow from the equation to the central ellipsoid in the form

$$Y_y{}^2\eta^2+Z_z{}^2\zeta^2+R^2(p\xi+q\eta+r\zeta)^2=1,$$

see Art. 34. By applying the usual analytical conditions that this is a surface of revolution we obtain the required relation between p, q, r.

Arrangement of Poinsot's central axes.

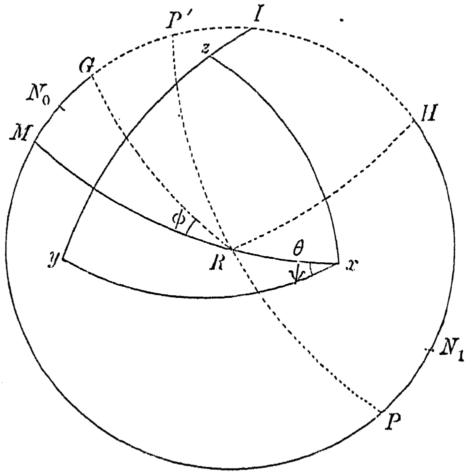
38. In whatever position the body is placed relatively to the forces it has been shown in Vol. I. that the forces acting on the body can be simplified into a single force, acting along a straight line called by Poinsot the central axis, and a couple round that axis. As the body takes different positions relative to the forces Poinsot's axis also moves relatively to both. In order to determine the arrangement of Poinsot's axes for all possible positions of the body and forces it will be convenient to have two systems of axes, one fixed in the body and the other fixed relatively to the forces.

Let the axes fixed in the body be the principal axes at the central point. These we shall represent by Ox, Oy, Oz. Following the same notation as before, the forces are represented by the astatic couples (G, b), (H, c), whose astatic arms are placed along y and z, together with a force R acting at O. The astatic moments of these couples are represented by K_2, K_3 respectively. Let the axes fixed in space be parallel to the force R, G, H.

These are represented by Ox', Oy', Oz'. We shall sometimes speak of them as the *axes of the forces*.

Let the direction cosines of either set of axes relatively to the other be given by the diagram. The positive direction of these axes are so chosen that by turning one set round the common origin the positive directions of x, y, z may be made to coincide with those of x', y', z'. The advantage of this choice is, that in the determinant of direction cosines every constituent is equal to its minor with the proper sign as given by the ordinary rules of determinants. Without losing the simplicity of the other relations of these constituents, we thus avoid any ambiguity of sign in the minors.

	x	y	z
R	a_1	a_2	a_3
G	b_1	b_2	b_3
H	c_1	c_2	c_3



In the figure the axes are represented in the manner usually adopted in spherical trigonometry. The axes Ox, Oy, Oz and Ox', Oy', Oz' cut the sphere in x, y, z and R, G, H respectively; the angles being represented by arcs of great circles. The Eulerian angular coordinates of R referred to x are $\theta = xR$, $\psi = yxR$, $\phi = MRG$. Since the angle between any two planes is equal to the arc joining their poles, it is easy to see that $zIG = \theta$, $Iz = \psi$, $IH = \phi$.

39. *To find the position of Poinsot's axis referred to the axes of the forces, and also the moment of the forces about it.*

Let Px'' be the required Poinsot's axis, Γ the moment of the

couple round it. The axis Px'' is parallel to Ox', let its coordinates referred to x', y', z', be η', ζ'.

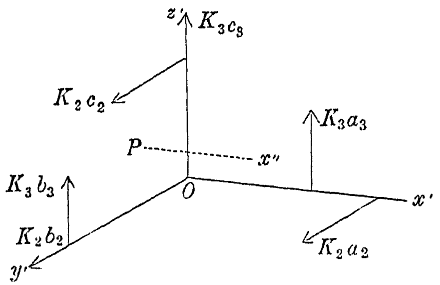
The couples K_2, K_3 have their astatic arms on the axes y, z, and their forces parallel to y', z'. To refer these couples to the axes x', y', z' we resolve the arms and move the forces parallel to themselves (Art. 6). Thus we replace the two couples by six others whose arms are arranged along the axes of x', y', z'. In the figure the forces at O are omitted to avoid complication, the arrows indicate the directions of the other forces of each of the six couples; and each arrow-head (as in Art. 10) is marked by the astatic moment of the corresponding couple.

By hypothesis all these couples together with a force R acting at O are equivalent to the couple Γ round Px'' and a force equal to R acting along Px''. Taking moments about the axes Ox', Oy', Oz' we have

$$\Gamma = K_3 b_3 - K_2 c_2 \qquad (1),$$

$$R\zeta' = -K_3 a_3 \qquad (2),$$

$$R\eta' = -K_2 a_2 \qquad (3).$$



Another proof. We may also obtain these results very simply without resolving the couples. Let the arms OB, OC of the couples be taken as unity so that the forces G, H acting at B and C are measured by the astatic moments K_2, K_3, Art. 10. The axes Ox', Oy', Oz' being the axes of reference, the coordinates of B and C are respectively a_2, b_2, c_2; a_3, b_3, c_3. Since K_2 acts parallel to Oy', its moment about Oz' is K_2a_2, and since K_3 acts parallel to Oz' its moment about Oy', is $-K_3a_3$. In the same way their moments about Ox' are K_3b_3 and $-K_2c_2$. Equating these to the moment of R acting along Px'' and of Γ we have the same results as before.

40. When the body is rotated about Ox', the direction cosines a_2, a_3 are invariable. It follows that the straight line whose position is determined by the equations (2) and (3) is fixed relatively to the forces. Hence we infer, that, *when the body is rotated about an axis passing through the central point and parallel*

to the principal force, Poinsot's axis always coincides with a straight line fixed in space.

This straight line traces out a right circular cylinder in the body whose radius ρ is given by the equation

$$R^2\rho^2 = K_2^2 a_2^2 + K_3^2 a_3^2 \dots\dots\dots\dots\dots\dots(4).$$

This cylinder is fixed in the body and moves with it. In one complete revolution of the body each generator in turn passes through the straight line fixed in space and becomes the Poinsot's axis for that position of the body.

Referring to the figure of Art. 38, the axis of this cylinder cuts the sphere of reference in R. We may also imagine the sphere of such size that the cylinder envelopes it along the circular boundary of the figure. In the figure the direction of the force R and the generators of the cylinder are supposed to be perpendicular to the plane of the paper.

As the body turns round OR as its axis, the dotted part of the figure remains fixed in space while the part indicated by the continuous lines moves round R.

Let a plane through the axis of the cylinder and the straight line fixed in space cut the sphere in the arc RP. Let RP produced backwards cut the circle GH in P'. Then the position of P or P' may be found from the equations

$$\tan GP = \tan GP' = \frac{\zeta'}{\eta'} = \frac{K_3 a_3}{K_2 a_2} \dots\dots\dots\dots(5).$$

In every position of the body Poinsot's central axis is a straight line drawn through P perpendicular to the plane of the circle GH. Here P is distinguished from P' by the sign of either η' or ζ' as given by the equations (2) and (3).

It follows from these results, that all the straight lines, each of which would be a Poinsot's axis if the body were properly placed, may be classified as the generators of a system of right circular cylinders. The axes of these cylinders pass through the central point and are always parallel to the direction of the principal force.

Conversely, *a straight line being given in the body, it may be required (when possible) to place the body in such a position that the straight line may be a Poinsot's axis.* To effect this, we turn the body about the central point until the given straight line is parallel to the principal force. If a_1, a_2, a_3 are the direction

cosines of the given straight line referred to the principal axes of the body at the central point, then, in this position of the body, a_1, a_2, a_3 are also the direction cosines of the principal force. If the distance of the given straight line from the central point does not satisfy equation (4) the straight line cannot be a Poinsot's axis. If however the equation is satisfied, we turn the body round the principal force as an axis of rotation through the angle GP determined by equation (5), or, which is the same thing, we turn the body until the given straight line passes through the point η', ζ' in the plane $y'z'$ determined by the equations (2), (3). The body has then been placed in the required position. When the straight line fixed in the body has been made parallel to the principal force the body may be inverted, so that the given straight line is again parallel to the force but points in the opposite direction. If the condition (4) is satisfied in one case, it is satisfied in the other. Thus if the construction yield one position in which the given straight line is a Poinsot's axis, it will yield another.

41. In every position of the body the couple-moment of Poinsot's axis is given by

$$\begin{aligned}\Gamma &= K_3 \cos Gz - K_2 \cos Hy \\ &= K_3 (\cos\psi \sin\phi + \sin\psi \cos\phi \cos\theta) \\ &\quad + K_2 (\sin\psi \cos\phi + \cos\psi \sin\phi \cos\theta),\end{aligned}$$

by using the spherical formulæ for the triangle GIz and HIy. This may be written in the form

$$\Gamma = \Gamma_0 \sin(\phi - \phi_0) \quad\quad (6),$$

where Γ_0 is the maximum value of Γ, and $\phi = \phi_0$ determines the position of the body when the couple-moment is zero. We easily find

$$\tan\phi_0 = -\frac{K_2 + K_3 \cos\theta}{K_2 \cos\theta + K_3} \tan\psi \quad\quad (7),$$

$$\left.\begin{aligned}\Gamma_0^2 &= (K_2 + K_3 \cos\theta)^2 \sin^2\psi + (K_2 \cos\theta + K_3)^2 \cos^2\psi \\ &= \frac{(K_2 + K_3 a_1)^2 a_3^2 + (K_2 a_1 + K_3)^2 a_2^2}{a_3^2 + a_2^2}\end{aligned}\right\} \quad (8).$$

Make the arc $MN_0 = \phi_0$, then the arc $N_0G = \phi - \phi_0$ and $\Gamma = \Gamma_0 \sin N_0G$. As the body rotates about the axis OR, both M and N_0 move with it. When $\phi - \phi_0 = 0$ or π, the point N_0 coincides with either P' or P; the couple-moment vanishes and the system is equivalent to a single resultant. As the body is turned from either of these opposite positions through any angle

the couple Γ increases and its magnitude varies as the sine of the angle of rotation. The couple reaches a maximum in either of the positions given by $\phi - \phi_0 = \pm \frac{1}{2}\pi$ and then decreases again. Thus there are in general two positions of the body in which the couple-moment Γ has a given value, and two more in which it has the same value with an opposite sign.

42. We may interpret this result in a slightly different manner. We may ascribe to each generator a certain couple-moment Γ peculiar to itself, which becomes the couple-moment when the body is so placed that that generator is a Poinsot's axis. Make $MN_1 = MN_0 + GP$, then for any generator of the cylinder say the one which passes through P we have $\Gamma = \Gamma_0 \sin N_1P$.

It will be useful to state this result in words. *Through the line of action of R draw two planes, one passing through the two generators whose couple-moments are each zero, and the other arbitrary and cutting the cylinders in two other generators. If Γ be the couple-moment for these last two generators and χ the angle between the planes, then $\Gamma = \Gamma_0 \sin \chi$ where Γ_0 is given by either of the forms in equation* (8).

43. In what precedes it has been supposed that both the direction and the line of action of the principal force R are given in the body. In this case the body can only be rotated about Ox' as an axis. If the direction of R is not given, but only its line of action, the body can also be inverted by rotating it through two right angles about an axis perpendicular to Ox'. To avoid complicating the figure it will be more convenient to effect this last change by rotating the forces in the opposite direction, each about its point of application, so that the angles between their directions remain unaltered.

The effect of this inversion is easily seen to be, that the positive directions of x' and of one of the two y', z' are reversed. As it will be convenient that they should have the same positive directions in space as before, we shall represent the effect of the inversion by changing the signs of the force R and of that of one of the astatic moments K_2, K_3. The sign of the couple-moment Γ about Poinsot's axis also must be changed (even if its magnitude remains unaltered) when the positive direction of x in space is to be the same after inversion as before.

One result of these changes is that the arc $P'P$ (Art. 40) takes up another position (say $Q'Q$, not drawn in the figure of Art. 38) making the same angle with GR as before, but on the other side. The angle ϕ_0 and the couple Γ_0 are also changed. Thus the positions in which Poinsot's couple vanishes are changed by the inversion of the body.

44. *To find the equation of Poinsot's axis referred to the principal axes at the central point.*

Following the notation already described in Art. 38, the

equations of Poinsot's axis referred to the axes of the forces are
$$R\eta' = -K_2a_2, \qquad R\zeta' = -K_3a_3 \quad\text{.............. (1)},$$
and the couple-moment Γ is given by
$$\Gamma = K_3b_3 - K_2c_2 \quad\text{...................... (2)}.$$
Transforming these to the axes fixed in the body, we obviously have
$$R(b_1\xi + b_2\eta + b_3\zeta) = -K_2a_2,$$
$$R(c_1\xi + c_2\eta + c_3\zeta) = -K_3a_3.$$
Eliminating ξ, η, ζ in turn, and remembering that each constituent of the determinant of transformation in Art. 38 is equal to its minor, we have
$$\left.\begin{aligned} R(-\eta a_3 + \zeta a_2) &= -K_2a_2c_1 + K_3a_3b_1 \\ R(-\zeta a_1 + \xi a_3) &= -K_2a_2c_2 + K_3a_3b_2 \\ R(-\xi a_2 + \eta a_1) &= -K_2a_2c_3 + K_3a_3b_3 \end{aligned}\right\} \quad\text{........... (3)}.$$

These may also be written in the form
$$\left.\begin{aligned} R(-\eta a_3 + \zeta a_2) - \Gamma a_1 &= -K_2b_3 + K_3c_2 \\ R(-\zeta a_1 + \xi a_3) - \Gamma a_2 &= \qquad\qquad -K_3c_1 \\ R(-\xi a_2 + \eta a_1) - \Gamma a_3 &= K_2b_1 \end{aligned}\right\} \quad\text{........ (4)}.$$
Any two of these are the equations to Poinsot's axis when the relative positions of the body and the forces are given by the direction cosines a_1, &c. They are also the equations of the fixed generator of the circular cylinder, Art. 40.

Adding together the squares of the equations (3), we obtain the equation of the cylinder traced out by Poinsot's axis as the body is turned round Ox'. This cylinder is easily seen to be a right circular cylinder and its radius ρ is given by
$$R^2\rho^2 = K_2^2a_2^2 + K_3^2a_3^2 \quad\text{.................... (5)},$$
as already proved in Art. 40.

When the body is so placed that the forces reduce to a single resultant, the equations (4) may be put into a more convenient form. Since $\Gamma = 0$, the first of those equations reduces to
$$\left.\begin{aligned} R(-\eta a_3 + \zeta a_2) &= -K_2b_3 + K_3c_2 \\ 0 &= -K_2c_2 + K_3b_3 \end{aligned}\right\}.$$

Subtracting the squares, we have
$$R^2(-\eta a_3 + \zeta a_2)^2 = (b_3^2 - c_2^2)(K_2^2 - K_3^2).$$
Let us seek the intersection of the single resultant with the plane of xy; putting therefore $\zeta = 0$, the two first of equations (4) become
$$R^2a_3^2\frac{\eta^2}{K_3^2 - K_2^2} = c_2^2 - b_3^2, \qquad R^2a_3^2\frac{\xi^2}{K_3^2} = c_1^2 \quad\text{...... (6)}.$$

A straight line drawn through the point thus determined parallel to the force R is the single resultant.

Adding these equations together and remembering that

$$b_3^2 + a_3^2 = 1 - c_3^2 = c_1^2 + c_2^2,$$

we have, after division by a_3^2,

$$\frac{\eta^2}{K_3^2 - K_2^2} + \frac{\xi^2}{K_3^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

This is the equation to a focal conic, Art. 36. The single resultant therefore intersects the focal conic in the plane of xy. In the same way, it intersects that in the plane of xz. We thus arrive at a *theorem due to Minding*, viz. that *when the body is so placed that the forces are equivalent to a single resultant, the line of action of that resultant is a focal line.* A further consideration of this mode of proof and of Minding's theorem will be found a little further on.

An apparent exception arises when either $a_3=0$ or $a_2=0$. Supposing that $a_3=0$ the equations (3) become $Ra_2\zeta = -K_2a_2c_1$, $Ra_1\zeta = K_2a_2c_2$.
Since $-c_2 = a_1b_3 - a_3b_1$, we have

$$\Gamma = K_3b_3 - K_2c_2 = (K_3 + K_2a_1)\, b_3 = 0.$$

Thus either $b_3=0$ or $K_3+K_2a_1=0$. Joining the former to $\Gamma=0$, we have $c_2=0$. The latter is impossible if K_3 is greater than K_2; if K_3 is less than K_2 the focal conic (7) is a hyperbola and the single resultant is parallel to an asymptote. Thus in both cases the single resultant intersects the focal conic.

Ex. 1. Show that the single resultant intersects the plane of the imaginary focal conic in the conic $$\frac{\eta^2}{K_2^2} + \frac{\zeta^2}{K_3^2} = \frac{1}{R^2}\left(\frac{1}{a_1^2} - 1\right).$$
This conic is fixed in the body when a_1 is given.

Ex. 2. Show that the circular cylinder (5) intersects the plane of xy in the conic whose equation is

$$R^2\{\xi^2 + \eta^2 - (\xi a_1 + \eta a_2)^2\} = K_2^2a_2^2 + K_3^2a_3^2.$$

45. The direction of the principal force R, and a point ξ, η, ζ on a generator of the circular cylinder being given referred to the principal axes of the body, it is required to find the couple-moment about that generator when the body is so placed that the generator is a Poinsot's axis.

For the sake of brevity let us write

$$-\eta a_3 + \zeta a_2 = p, \quad -\zeta a_1 + \xi a_3 = q, \quad -\xi a_2 + \eta a_1 = r.$$

Multiplying the second and third of equations (4) Art. 44 by $K_2^2a_2$ and $K_3^2a_3$ respectively we have

$$K_2^2a_2(Rq - \Gamma a_2) + K_3^2a_3(Rr - \Gamma a_3) = K_2K_3(-K_2a_2c_1 + K_3a_3b_1) = K_2K_3Rp.$$

The couple-moment Γ is therefore given by

$$(K_2^2a_2^2 + K_3^2a_3^2)\,\Gamma = R\,(K_2^2a_2q + K_3^2a_3r - K_2K_3p)\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If the line of action of R only is given and the force may act either way along it, we obtain another value of Γ by inverting either the body or the forces. If Γ' be the couple-moment after inversion we have by Art. 43

$$(K_2^2a_2^2 + K_3^2a_3^2)\,\Gamma' = R\,(K_2^2a_2q + K_3^2a_3r + K_2K_3p)\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The force R then acts along the negative direction of its line of action.

We may write (1) in the form

$$(K_2^2a_2^2+K_3^2a_3^2)\,\Gamma=R\,\{-(K_3^2-K_2^2)\,a_2a_3\xi+K_3(K_3a_1+K_2)a_3\eta-K_2(K_2a_1+K_3)\,a_2\zeta\}\ldots(3).$$

We therefore see that the plane through the line of action of R and the two generators whose couple-moments are zero (Art. 41) is

$$-(K_3^2-K_2^2)\,a_2a_3\xi+K_3\,(K_3a_1+K_2)\,a_3\eta-K_2\,(K_2a_1+K_3)\,a_2\zeta=0\ldots\ldots\ldots(4).$$

Conversely, *when the magnitude of the couple* Γ *is given, either of the equations* (1) *or* (3) *enables us to find the generators which have the given moment* Γ *when the body is so placed that one of them is a Poinsot's axis.* When Γ is given, either of these equations represents a plane intersecting the circular cylinder (5) in two straight lines which are parallel to the principal force. These are the generators required; see also Art. 41. If we change the sign of Γ we obtain another plane, parallel to the former, giving two other generators, each of whose couple moments have the given magnitude but an opposite sign. These four are obviously symmetrically arranged round the principal force.

Another construction for Poinsot's axis and moment is indicated in the following examples.

Ex. 1. A straight line OQ is drawn through the central point O perpendicular to the plane containing the force R and its corresponding fixed generator. Prove that p, q, r are the coordinates of the point Q in which this straight line cuts the circular cylinder. Prove also that Q is one of the poles of the great circle represented by PP' in the figure of Art. 38.

Ex. 2. Let OS be the straight line whose direction cosines are proportional to $-K_2K_3$, $K_2^2a_2$, $K_3^2a_3$, when referred to the principal axes of the body at the central point O; thus OS is fixed in the body when the position of OR is given. If ϕ be the angle contained by the lines OQ, OS, prove that

$$\frac{\Gamma}{\cos\phi}=\left\{\frac{K_2^2K_3^2+K_2^4a_2^2+K_3^4a_3^2}{K_2^2a_2^2+K_3^2a_3^2}\right\}^{\frac{1}{2}}.$$

Show also that the straight line OS lies in the plane containing the force R and the two generators whose couple-moments are zero.

46. If the magnitude of the couple-moment Γ is given as well as the line of action of R, we may obtain other cylinders which will intersect the right cylinder already found in the corresponding Poinsot's axes.

The first of equations (4) Art. 44 is

$$R\,(-\eta a_3+\zeta a_2)-\Gamma a_1=-K_2b_3+K_3c_2,$$

and

$$\Gamma=-K_2c_2+K_3b_3.$$

Hence subtracting the squares, as in Art. 44,

$$\{R\,(-\eta a_3+\zeta a_2)-\Gamma a_1\}^2-\Gamma^2=(b_3^2-c_2^2)\,(K_2^2-K_3^2).$$

Now by Art. 38 $b_3^2-c_2^2=c_1^2-a_3^2$, hence, substituting for c_1^2 from the second of equations (4), we have

$$\frac{\{R\,(-\eta a_3+\zeta a_2)-\Gamma a_1\}^2-\Gamma^2}{K_3^2-K_2^2}+\frac{\{R\,(-\zeta a_1+\xi a_3)-\Gamma a_2\}^2}{K_3^2}=a_3^2\ \ldots\ldots\ldots(1).$$

Again $b_3^2-c_2^2=a_2^2-b_1^2$, substituting for b_1^2 from the third of equations (4), we have

$$-\frac{\{R\,(-\eta a_3+\zeta a_1\}^2-\Gamma a_1\}^2-\Gamma^2}{K_3^2-K_2^2}+\frac{\{R\,(-\xi a_2+\eta a_1)-\Gamma a_3\}^2}{K_2^2}=a_2^2\ \ldots\ldots\ (2).$$

Lastly, the last two of equations (4) give

$$\frac{\{R(-\zeta a_1+\xi a_3)-\Gamma a_2\}^2}{K_3^2}+\frac{\{R(-\xi a_2+\eta a_1)-\Gamma a_3\}^2}{K_2^2}=1-a_1^2 \text{ .... } (3).$$

The three surfaces (1) (2) and (3) are cylinders, for the equation to any one of them shows that an expression of the first degree in ξ, η, ζ is some function of another expression of the first degree. Also the axis of each cylinder is parallel to the straight line $\xi/a_1=\eta/a_2=\zeta/a_3$, i.e. the axis of each is parallel to the line of action of the force R.

It may be noticed that the direction cosines b_1, b_2, b_3; c_1, c_2, c_3 have been eliminated so that the equations to these cylinders contain only the principal force R, the direction cosines of R and Poinsot's couple Γ.

47. Supposing that the coordinates (ξ, η, ζ) of some point on the cylindrical locus (5) are given, and that the line of action of the force R is also known, any one of the equations (1), (2), (3), of Art. 46 may be regarded as a quadratic to find the couple-moment when the body is so placed that the corresponding generator is a Poinsot's axis.

If we seek the corresponding equations when the forces are inverted we change the signs of R, Γ and one of the K's (Art. 43). But these changes leave the quadratics unaltered. Thus the two values of Γ given by any one of these quadratics correspond to the two directions in which R can act along the same given line of action.

Ex. The given point (ξ, η, ζ) being supposed to be on the circular cylinder, prove that the three quadratics (1) (2) (3) of Art. 46 reduce to the same, viz.

$$\Gamma^2(K_2^2a_2^2+K_3^2a_3^2)-2R\Gamma(K_2^2a_2q+K_3^2a_3r)+R^2(K_2^2q^2+K_3^2r^2)=K_2^2K_3^2(a_2^2+a_3^2).$$

Prove also that the roots of this quadratic are given by

$$\Gamma(K_2^2a_2^2+K_3^2a_3^2)=R(K_2^2a_2q+K_3^2a_3r\mp K_2K_3p)$$

where p, q, r have the meanings specified in Art. 45.

48. **Minding's Theorem.** By joining any one of the three cylinders (1), (2), (3) to the circular cylinder we have sufficient equations to find the generators which can have a given couple-moment and are also parallel to any given straight line. It will often be more convenient to use the intersections of these cylinders with one of the coordinate planes. Thus putting $\zeta=0$, the cylinder (1) cuts the plane of xy in the conic

$$\frac{(R\eta a_3+\Gamma a_1)^2-\Gamma^2}{K_3^2-K_2^2}+\frac{(R\xi a_3-\Gamma a_2)^2}{K_3^2}=a_3^2 \text{ .... } (1).$$

When the forces are equivalent to a single resultant we have $\Gamma=0$ and in that case equation (1) reduces to the focal conic

$$\frac{\eta^2}{K_3^2-K_2^2}+\frac{\xi^2}{K_3^2}=\frac{1}{R^2} \text{ .... } (2).$$

The single resultant therefore intersects the focal conic in the plane of xy. Similarly it intersects that in the plane of xz. See Art. 44.

49. *Conversely*, let a straight line intersect both focal conics, then by Art. 37 it is a generator of the circular cylinder. If the direction cosines of this straight line are a_1, a_2, a_3, the corresponding couple-moment Γ is given by the quadratic (1) of Art. 48.

This quadratic gives two values of Γ. Multiplying (2) by $R^2a_3^2$ and subtracting the result from (1) we find that one root is $\Gamma=0$ and that the other is given by

$$(K_2^2a_2^2+K_3^2a_3^2)\Gamma=2Ra_3\{K_2^2\xi a_2+K_3^2(\eta a_1-\xi a_2)\} \text{ .... } (3).$$

The result is that the couple-moment for the generator is zero for one of the two directions in which the force R can act along that generator.

These two values of Γ follow also from equations (1) and (2) Art. 45 for when the value of Γ given by (1) is zero, the value given by (2) agrees with that shown in equation (3) of this article.

Finally, we see that *if any straight line can be the line of action of a single resultant force that line must intersect both the focal conics, and if a straight line intersect both the focal conics it can be the line of action of a single resultant if the body be properly placed.*

50. Ex. 1. The direction of the principal force R being given by the direction cosines a_1, a_2, a_3 referred to the principal axes at the central point show that each of the planes
$$\left(\frac{\xi}{a_1}-\frac{\eta}{a_2}\right)K_3{}^2 \pm \left(\frac{\eta}{a_2}-\frac{\zeta}{a_3}\right)\frac{K_2K_3}{a_1}+\left(\frac{\zeta}{a_3}-\frac{\xi}{a_1}\right)K_2{}^2=0$$
passes through the line of action of R and intersects the focal conics in four points, which are the corners of a parallelogram formed by the focal lines, two of which are parallel to the direction of R. Prove also that the focal lines parallel to the given direction of R are the corresponding single resultants.

This follows easily from Art. 45.

Ex. 2. If the body is so placed that the force R acts along an asymptote of the hyperbolic focal conic, prove (1) that the circular cylinder contains the elliptic focal conic on its surface; (2) that as the body is turned round OR Poinsot's axis lies in the plane containing R and parallel to the force H which corresponds to the greater astatic moment K_3; (3) that Poinsot's couple Γ is always zero as the body is turned found OR provided the force R acts in the proper direction, but is zero only when the plane of the hyperbolic conic contains the force H if R act in the other direction.

51. **Relations of Poinsot's axis to the confocals.** The manner in which the single resultant is connected with the confocals is given by Minding's theorem. We may also find the relations of Poinsot's axis with the same confocals in the general case in which the couple is not zero. To effect this we require the following lemma in solid geometry.

52. *Lemma.* Let the squares of the semi-axes of two confocals be $\alpha^2+\lambda$, $\beta^2+\lambda$, $\gamma^2+\lambda$ and $\alpha^2+\lambda'$, $\beta^2+\lambda'$, $\gamma^2+\lambda'$. Let the direction cosines of any straight line be (l, m, n) and its distance from the origin be ρ. If two planes at right angles can be drawn through the straight line to touch the two confocals, then
$$\rho^2+\alpha^2l^2+\beta^2m^2+\gamma^2n^2=\alpha^2+\beta^2+\gamma^2+\lambda+\lambda'.$$
It follows that when the confocals are given the left-hand side is constant for all straight lines.

Let (l', m', n'), (l'', m'', n'') be the direction cosines of the tangent planes, and p, p' the lengths of the perpendiculars on them. Then
$$p^2=(\alpha^2+\lambda)\,l'^2+(\beta^2+\lambda)\,m'^2+(\gamma^2+\lambda)\,n'^2,$$
$$p'^2=(\alpha^2+\lambda')\,l''^2+(\beta^2+\lambda')\,m''^2+(\gamma^2+\lambda')\,n''^2.$$
Noticing that $\rho^2=p^2+p'^2$ we find by addition
$$\rho^2=\alpha^2\,(l'^2+l''^2)+\beta^2\,(m'^2+m''^2)+\gamma^2\,(n'^2+n''^2)+\lambda+\lambda'.$$
Hence since $l^2+l'^2+l''^2=1$ &c., we have
$$\rho^2+\alpha^2l^2+\beta^2m^2+\gamma^2n^2=\alpha^2+\beta^2+\gamma^2+\lambda+\lambda'.$$

53. Let us now apply this Lemma to any generator of the cylinder. Let α, β, γ be the semi-axes of the imaginary focal conic, then, by Art. 36,
$$\alpha^2=0,\quad \beta^2=-K_2{}^2/R^2,\quad \gamma^2=-K_3{}^2/R^2.$$

The values of λ, λ' are the squares of the semi-major axes of the two confocals; let these be represented by M_1/R^2 and M_1'/R^2 as in Art. 35. The direction cosines of any generator are (a_1, a_2, a_3) and its distance ρ from the central point is given by $R^2\rho^2 = K_2^2a_2^2 + K_3^2a_3^2$. Hence, substituting the left-hand side of the equation in the Lemma reduces to zero. We therefore have $M_1^2 + M_1'^2 = K_2^2 + K_3^2$.

If therefore any two planes at right angles are drawn through a possible Poinsot's axis and two confocals are drawn to touch these planes, the sum of the squares of the semimajor axes of these confocals is constant. This constant when multiplied by R^2 is the sum of the squares of the astatic moments of the principal couples at the central point.

From this we may deduce as a corollary a theorem discovered by Darboux.

Let a plane be drawn through any possible Poinsot's axis to touch one of the focal conics, then a perpendicular plane through the same axis will touch another focal conic.

For in the limit these conics may be regarded as the bounding rims of two flat confocals whose semi-major axes are respectively K_2/R and K_3/R.

54. Ex. 1. If a possible Poinsot's axis touch two confocals prove that the sum of the squares of their semi-major axes is equal to $K_2^2 + K_3^2$ after division by R^2.

If a straight line touch two confocals, and tangent planes are drawn at the points of contact, these planes are known to be at right angles. If we apply the general theorem in Art. 53 to these two tangent planes, the result follows at once.

Ex. 2. If a possible Poinsot's axis intersect one of the focal conics prove that it must intersect the other also.

For suppose it intersects the plane of xy in the elliptic focal conic, it may be regarded as touching the confocal surface whose semi-major axis is K_3/R. Hence it also touches the confocal surface whose semi-major axis is K_2/R (by the last example), i.e. it intersects the plane of xz in the hyperbolic focal conic.

Reduction to Three and to Four Forces.

55. We have seen that the forces of any astatic system may be reduced to two couples and a single force. This representation of the force, though very simple in its character, may not always be convenient. These couples and the force have an intimate relation to the central point and central plane, and the positions of this point and plane may not suit the circumstances of the problem we wish to consider.

We shall now examine some other representations of an astatic system. We shall show that the forces may be reduced to *three forces* which act at three arbitrary points in the central plane. These points however must not in general lie in one straight line. We shall show that the forces of the system may also be reduced to *four forces* which act at any four points fixed in the body at which we may find it convenient to apply them. The four points must not in general lie in one plane.

We can see another advantage of these representations of the forces. For the points of application may be regarded as the

corners of a triangle or tetrahedron of reference. We are thus enabled to use the systems of coordinates called trilinear and tetrahedral with considerable effect.

56. *To show that all the forces of any system may be reduced to three forces which act at three points lying in the central plane.*

Following the same notation as in Art. 9, let the forces of the system be P_1, P_2, &c. and let M_1, M_2,... be their points of application. Let these be referred to any axes Ox, Oy, Oz, either rectangular or oblique, which are fixed relatively to the body. Let the coordinates of M_1, M_2, &c. be (x_1, y_1, z_1), (x_2, y_2, z_2), &c. Let Ox', Oy', Oz', be another system of axes, not necessarily rectangular, to which we may refer the forces. These are fixed relatively to the forces. Let the components of the forces along these be $(X'_1, Y'_1, Z'_1,)$, $(X'_2, Y'_2, Z'_2,)$, &c.

Consider the system of parallel forces X'_1, X'_2, &c. All these are astatically equivalent to a single force $\Sigma X'$ acting at their centre of parallel forces. In the same way the two other systems of parallel forces viz. Y'_1, Y'_2 &c. and Z'_1, Z'_2 &c. are equivalent to $\Sigma Y'$ and $\Sigma Z'$ each acting at its own centre of parallel forces in directions parallel to y' and z' respectively. These forces we may represent by F, G, H, and their points of application by A, B, C. The centre of parallel forces is known to possess the astatic quality. If then we move the arbitrary axes Ox', Oy', Oz' in any manner about the origin, keeping their inclination to each other unaltered, *the system will yet be equivalent to the same three forces F, G, H acting at the same three points A, B, C in directions always parallel to the axes Ox', Oy', Oz'.*

To find the coordinates of these points we may therefore consider any one position of the forces and the body. In this position let X, Y, Z be the components of any force P resolved along the axes Ox, Oy, Oz. Then

$$X' = lX + l'Y + l''Z, \quad Y' = mX + m'Y + m''Z \quad Z' = \&c.$$

where (l, m, n), (l', m', n'), (l'', m'', n''), are the direction ratios of the axes (x, y, z) referred to (x', y', z').

Let $(\bar{x}, \bar{y}, \bar{z},)$ be the coordinates of A, then

$$\bar{x}_1 = \frac{\Sigma X'x}{\Sigma X'} = \frac{l\Sigma Xx + l'\Sigma Yx + l''\Sigma Zx}{l\Sigma X + l'\Sigma Y + l''\Sigma Z}$$

with similar values for $\bar{y}_1$ and $\bar{z}_1$. Taking the same notation as in Art. 10 we write $\Sigma Xx = X_x$ &c. $\Sigma X = X_0$ &c. We thus

have
$$\left.\begin{aligned} F\bar{x}_1 &= lX_x + l'Y_x + l''Z_x \\ F\bar{y}_1 &= lX_y + l'Y_y + l''Z_y \\ F\bar{z}_1 &= lX_z + l'Y_z + l''Z_z \\ F &= lX_0 + l'Y_0 + l''Z_0 \end{aligned}\right\} \quad \text{.................(1).}$$

Hence it appears that the point A lies on the plane

$$\begin{vmatrix} \xi & X_x & Y_x & Z_x \\ \eta & X_y & Y_y & Z_y \\ \zeta & X_z & Y_z & Z_z \\ 1 & X_0 & Y_0 & Z_0 \end{vmatrix} = 0 \quad \text{....................(2).}$$

In the same way the points B and C also lie on this plane.

57. We notice that the directions of the axes Ox', Oy', Oz', are perfectly arbitrary except that they cannot all lie in one plane. We may therefore obtain an infinite variety of triangles ABC with corresponding forces at the corner. Any one of these may be called an *astatic triangle,* and the points A, B, C, may be called *astatic points.*

We may obviously make the inclinations of the forces F, G, H to each other whatever we please, though of course the position of the triangle ABC is dependent on our choice of these inclinations. It is generally most convenient to make the forces F, G, H act in directions at right angles to each other.

We have seen that when we want to find the positions of A, B, C we may consider the body to have some fixed position relative to the forces. For this position X_x &c. are all constant whatever the positions of the axes x', y', z' may be. The equation (2) therefore gives, as the locus of the points A, B, C, a plane fixed in the body. We also see that the locus is a unique plane except when all the coefficients are zero. An independent and elementary proof that the plane ABC is unique has been given in Art. 13.

Comparing the equation (2) with that found in Art. 25 we notice that this plane is the same as that already called the *central plane.* It follows that all the astatic triangles lie on the central plane.

58. *To find the central plane and one astatic triangle with rectangular forces.*

The theorem proved in Art. 56 supplies us with a useful method of finding the position of the central plane. To effect this we resolve all the forces of the system into any three direc-

tions we may find convenient. Taking the forces in these three directions separately we have three sets of parallel forces. We then find the centre of parallel forces of each set by any method we may find convenient. We thus arrive at three points which we call A, B, C. The plane through A, B, C is the central plane. We have also found one astatic triangle.

Suppose the system referred to rectangular axes Ox, Oy, Oz and consider any position of the body relative to the forces. All the x-components form a system of parallel forces which may be collected into a single astatic force $\Sigma X = F$ acting at a point A whose coordinates are

$$\bar{x}_1 = \frac{\Sigma Xx}{\Sigma X} \qquad \bar{y}_1 = \frac{\Sigma Xy}{\Sigma X} \qquad \bar{z}_1 = \frac{\Sigma Xz}{\Sigma X}.$$

In the same way the y-components may be collected into a force $\Sigma Y = G$ acting at a point B whose coordinates are

$$\bar{x}_2 = \frac{\Sigma Yx}{\Sigma Y} \qquad \bar{y}_2 = \frac{\Sigma Yy}{\Sigma Y} \qquad \bar{z}_2 = \frac{\Sigma Yz}{\Sigma Y}.$$

The z-components may be similarly treated.

These three points lie on the central plane. The forces F, G, H act in directions at right angles at each other and their magnitudes have been found.

If the principal force is finite, the axes may always be so chosen that ΣX, ΣY, ΣZ are not zero. If the principal force is zero, the coordinates of the three points are either infinite or take an indeterminate form; and in this case the central plane is either at an infinite distance or is indeterminate in position. Thus whenever there is a central plane this construction may be used to find it.

59. Referring to the table of elementary couples given in Art. 10 these expressions for $(\bar{x}, \bar{y}, \bar{z})$ &c. give a new interpretation to those symbols. It has been shown in Art. 10 that the *constituents in any row* of that table are the components of the corresponding couples. It has now been proved that the *constituents in any column* are proportional to the coordinates of an astatic triangle with rectangular forces.

60. *To reduce all the forces of any system to four forces which act at four given points not all in one plane.*

Let A, B, C, D be any four points fixed in the body. These we shall regard as the corners of the tetrahedron of reference.

Let P_1, P_2, &c., be any forces acting on the body and let M_1, M_2, &c. be their points of application. We propose to replace each of these by four forces acting at the corners A, B, C, D parallel to the original direction of the force. Consider DA, DB, DC to be a system of oblique axes, let ξ, η, ζ, be the coordinates of any point M and let $DA = a$, $DB = b$, $DC = c$. Then by Art. 7 the forces acting at A, B, C, D are respectively

$$P\xi/a, \quad P\eta/b, \quad P\zeta/c, \quad P - P\xi/a - P\eta/b - P\zeta/c.$$

Now ζ/c is equal to the ratio of the perpendiculars drawn from M and C on the face ABC, and this ratio is the tetrahedral coordinate of M. Representing the four *tetrahedral* coordinates of M by α, β, γ, δ, and remembering that their sum is unity we see that the four forces at the corners A, B, C, D, are respectively $P\alpha$, $P\beta$, $P\gamma$, $P\delta$.

We therefore have the following working rule. *Any force P acting at the point whose tetrahedral coordinates are α, β, γ, δ may be replaced by four parallel forces acting at the corners of the tetrahedron of reference whose magnitudes are respectively $P\alpha$, $P\beta$, $P\gamma$, $P\delta$.*

The several forces acting at each corner may now be compounded together. The result is that any system of forces can be replaced by four forces, one at each corner of the tetrahedron.

61. We may prove in the same way that a force P acting at any point M in the plane ABC may be replaced by three parallel forces respectively equal to $P\alpha$, $P\beta$, $P\gamma$, and acting at A, B, C, where α, β, γ, are the areal coordinates of M referred to the triangle ABC.

We may also deduce this result from the general theorem for a tetrahedron. We notice that tetrahedral coordinates become areal when the point considered lies in a coordinate plane. We may therefore disregard the coordinate δ and treat the tetrahedral coordinates α, β, γ, as if they were areal.

62. *To show that the system can be reduced to three forces acting at any three points in the central plane which form a triangle.*

Let the system be reduced to three forces acting at the corners A, B, C of some astatic triangle; then this triangle lies in the central plane. Let A', B', C', be any three points in the same plane, but not in a straight line, and let D' be a fourth point not

in that plane. Regarding $A'B'C'D'$ as the tetrahedron of reference we shall transfer the forces from A, B, C to the corners of this tetrahedron.

To find the force at D', we multiply each force by its δ coordinate. Since this coordinate is zero for each of the points A, B, C, the resultant force at D' is zero.

63. **Transformation of Triangles.** *One astatic triangle ABC and the rectangular forces F, G, H at its corners being given, it is required to transfer this representation to any other triangle $A'B'C'$ and to find the rectangular forces F', G', H' at its corners.*

Let axes drawn through any point O parallel to either of these sets of forces be called the axes of those forces. We thus have two sets of rectangular axes. Let their mutual direction cosines be given in the usual way by the diagram.

	F'	G'	H'
F	l	m	n
G	l'	m'	n'
H	l''	m''	n''

Then any force F may be resolved into the components Fl, Fm, Fn, acting respectively parallel to the axes of F', G', H'. Treating the forces G, H in the same way we have $F' = Fl + Gl' + Hl''$

$$G' = Fm + Gm' + Hm'', \qquad H' = \&c.$$

We also have

$$F = F'l + G'm + H'n, \quad G = F'l' + G'm' + H'n', \quad H = \&c.$$

The point of application of the force F' is the centre of the parallel forces Fl, Gl', Hl'' which act at A, B, C. Thus the point A' at which F' acts is the centre of gravity of three weights (positive or negative) proportional to Fl, Gl', Hl'' placed at the corners A, B, C of the given triangle. By properly choosing these ratios we can place the corner A' at any point we please.

The areal coordinates of the corners of either triangle referred to the other can also be found very simply by using the theorem of Art. 61. Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ be the *areal* coordinates of the points A', B', C' referred to the given triangle ABC. If we transfer the forces F', G', H' back again to the triangle ABC, the three forces at A will be $F'\alpha_1$, $G'\alpha_2$, $H'\alpha_3$. But these are the components of F. The forces at B, C, may be similarly found.

Hence

$$\begin{array}{lll} F'\alpha_1 = Fl & F'\beta_1 = Gl' & F'\gamma_1 = Hl''. \\ G'\alpha_2 = Fm & G'\beta_2 = Gm' & G'\gamma_2 = Hm'', \\ H'\alpha_3 = Fn & H'\beta_3 = Gn' & H'\gamma_3 = Hn''. \end{array}$$

By choosing the nine direction cosines in any way which their

mutual relations permit we can use these formulæ to transform from one triangle to another.

If the forces of the two triangles are oblique we regard (l, m, n), (l', m', n') (l'', m'', n''), as the direction ratios of F, G, H referred to the axes F', G', H'. The direction ratios of F', G', H' referred to the axes of F, G, H, are proportional to the minors of (l, l', l'') &c. If these direction ratios be $(\lambda, \lambda', \lambda'')$ (μ, μ', μ'') (ν, ν', ν'') we have

$$F=F'\lambda+G'\mu+H'\nu,\quad G=\&c.,\quad H=\&c.,$$

instead of the expressions given above. With this exception all the other equations in this article apply to oblique forces.

64. **The imaginary focal Conic.** Let us suppose that the forces of the two triangles ABC, $A'B'C'$ are rectangular. The nine direction cosines are connected by relations such as $lm+l'm'+l''m''=0$ &c. Hence the coordinates of A', B', C' are connected by the three equations

$$\left.\begin{aligned}\frac{\alpha_1\alpha_2}{F^2}+\frac{\beta_1\beta_2}{G^2}+\frac{\gamma_1\gamma_2}{H^2}&=0\\ \frac{\alpha_2\alpha_3}{F^2}+\frac{\beta_2\beta_3}{G^2}+\frac{\gamma_2\gamma_3}{H^2}&=0\\ \frac{\alpha_3\alpha_1}{F^2}+\frac{\beta_3\beta_1}{G^2}+\frac{\gamma_3\gamma_1}{H^2}&=0\end{aligned}\right\}\ldots\ldots\ldots(1).$$

If therefore A' be taken at any point $(\alpha, \beta, \gamma,)$ both B' and C' must lie on the straight line

$$\frac{\alpha_1\alpha}{F^2}+\frac{\beta_1\beta}{G^2}+\frac{\gamma_1\gamma}{H^2}=0\ldots\ldots\ldots(2).$$

where α, β, γ are current coordinates. Taking B' anywhere on this line, then C' is found as the intersection of two straight lines.

This straight line (2) is evidently the polar line of $(\alpha, \beta, \gamma,)$ with regard to the imaginary conic

$$\frac{\alpha^2}{F^2}+\frac{\beta^2}{G^2}+\frac{\gamma^2}{H^2}=0\ldots\ldots\ldots(3).$$

Thus the three astatic points are always at the corners of a self-conjugate triangle with regard to this conic.

The statical property of this conic is that each side of every astatic triangle with rectangular forces is the polar line of the opposite corner. But as two different conics cannot have the polar lines of every point the same in each conic, it follows that this conic is unique. Whatever astatic triangle ABC we take as the triangle of reference, the conic given by this equation is the same.

65. Ex. 1. Show that, whatever astatic triangle with rectangular forces is taken as the triangle of reference, the quantities

(1) $F^2+G^2+H^2$,

(2) $FGH\Delta$,

(3) $a^2G^2H^2+b^2H^2F^2+c^2F^2G^2$,

are invariable, where a, b, c are the sides, Δ the area of the triangle, and F, G, H the forces.

We have also the invariant property that the centre of gravity of three weights, proportional to F^2, G^2, H^2, placed at the corners is the same for all triangles.

Ex. 2. Show that, whatever astatic triangle with oblique forces is taken as the triangle of reference, the quantities

(1) $F^2+G^2+H^2+2FG\cos\gamma+2GH\cos\alpha+2HF\cos\beta$

(2) $FGH\Delta\mu$

(3) $a'^2G'H'\{F'^2(\cos\alpha-\cos\beta\cos\gamma)-F'H'(\cos\beta-\cos\gamma\cos\alpha)$
$$-F'G'(\cos\gamma-\cos\alpha\cos\beta)-G'H'\sin^2\alpha\}+\&c.+\&c.$$

are invariable, where α, β, γ are the mutual inclinations of the forces and

$$\mu=1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma.$$

We notice that μ is six times the volume of the tetrahedron formed by unit lines drawn from any point parallel to the forces. It follows that μ cannot vanish unless the astatic forces are parallel to one plane.

Ex. 3. A system of forces is equivalent to a force R, acting at a point O, and two couples, whose astatic moments are K_2, K_3, and whose astatic arms are placed along the rectangular axes OY, OZ, the forces of the couples being perpendicular to each other and to the force R, see Art. 32. If these are transferred to an astatic triangle $A'B'C'$ situated in the plane yz, the coordinates of the corners being (η_1, ζ_1), (η_2, ζ_2), (η_3, ζ_3) and the rectangular forces F', G', H', prove that

$$\begin{array}{lll} F'=Rl & F'\eta_1=K_2l' & F'\zeta_1=K_3l'' \\ G'=Rm & G'\eta_2=K_2m' & G'\zeta_2=K_3m'' \\ H'=Rn & H'\eta_3=K_2n' & H'\zeta_3=K_3n'' \end{array}$$

where l, m, n &c. are the nine direction cosines of F', G', H', as in Art. 63.

If the forces F', G', H' are all equal, prove that the sum of the distances of the three corners from each of the axes of y and z is zero.

66. **To find the Central Point.** *The astatic triangle A, B, C, with rectangular forces F, G, H being given, show that the central point is the centre of gravity of three weights proportional to F^2, G^2, H^2 placed at the corners.*

This follows easily from the theorem proved in Art. 30. We multiply each force, such as F, by the resolved part of all the forces along it, i.e. by F; the product is F^2. The rule asserts that the central point is the centre of gravity of the three products F^2, G^2, H^2, placed at the points of application of F, G, H.

Ex. If the forces F, G, H of an astatic triangle are not rectangular prove that the central point is the centre of gravity of three weights proportional to

$$F(F+G\cos\gamma+H\cos\beta),\quad G(F\cos\gamma+G+H\cos\alpha)\quad H(F\cos\beta+G\cos\alpha+H)$$

placed at the corners, where α, β, γ are the angles between the forces (G, H), (H, F), (F, G).

This result follows at once from the general theorem given in Art. 30.

67. *The central point coincides with the centre of the imaginary conic.* To find the centre of the conic we follow the rule given in treatises on Conics. Differentiating the equation of the conic (Art. 64) with regard to the areal coordinates α, β, γ separately, and equating the results, we find that α, β, γ are proportional to F^2, G^2, H^2. The result follows at once.

68. *The imaginary conic being given, it is required to find the central lines and the principal moments of the system.*

Let the system of forces be reduced to its simplest form (Art. 32), i.e. let the forces be represented by a force R acting at the central point O together with two astatic couples whose arms are placed along the central lines Oy, Oz. Let the astatic moments be K_2, K_3.

Consider the origin O as one corner of an astatic triangle and produce the arms

of the couples to very distant points B and C, replacing the forces by two others, viz. G and H both very small. Then OBC is an infinitely large astatic triangle with *rectangular forces*. Let $OB=b$, $OC=c$ then $bG=K_2$ and $cH=K_3$ also $F=R$. We shall now use this triangle to find the equation to the imaginary conic by the formula given in Art. 64.

Let η, ζ be the Cartesian coordinates referred to the rectangular axes Oy, Oz of any point. Let α, β, γ be the areal coordinates of the same point referred to the infinitely large triangle OBC. Then $\alpha=1$, $\beta=\eta/b$, $\gamma=\zeta/c$. The conic

$$\frac{\alpha^2}{F^2}+\frac{\beta^2}{G^2}+\frac{\gamma^2}{H^2}=0$$

therefore reduces to

$$\frac{\eta^2}{K_2^2}+\frac{\zeta^2}{K_3^2}+\frac{1}{R^2}=0.$$

We therefore infer (1) that the centre of the imaginary conic is the central point, (2) the principal diameters are the central lines of the system, (3) that the lengths of the principal semidiameters are $K_2\sqrt{-1}/R$ and $K_3\sqrt{-1}/R$.

Referring to Art. 36, we see that the imaginary conic is the same as the imaginary focal conic.

69. Ex. 1. If ABC be an astatic triangle with rectangular forces show that either central line makes an angle θ with the side BC where

$$a\tan 2\theta=\frac{4\Delta F^2(H^2b\cos C-G^2c\cos B)}{a^2G^2H^2+H^2F^2b^2\cos 2C+F^2G^2c^2\cos 2B},$$

and Δ is the area of the triangle.

Ex. 2. If a triangle having its orthocentre at the central point be projected orthogonally on the central plane, prove that the projection is a possible astatic triangle with rectangular forces, provided the self-conjugate circle projects into the real conic

$$\frac{\eta^2}{K_2^2}+\frac{\zeta^2}{K_3^2}=\frac{1}{R^2}.$$

70. **Transformation of tetrahedra.** The forces being referred to one tetrahedron as $ABCD$, it is required to refer them to any other tetrahedron as $A'B'C'D'$.

If the coordinates of the corners of the first tetrahedron with regard to the second are known, the transference may be effected at once by using the rule given in Art. 60. But if the coordinates of the second tetrahedron with regard to the first are given, we may proceed in the following manner.

Let the tetrahedral coordinates of $A'B'C'D'$ referred to the first tetrahedron be given by the diagram, and let the whole determinant be Δ. Then the coordinates of A referred to the second tetrahedron are the minors of the several terms in the row opposite A after division by Δ. The coordinates of B are the minors of the terms in the row opposite B after division by Δ, and so on.

	A'	B'	C'	D'
A	a_1	b_1	c_1	d_1
B	a_2	b_2	c_2	d_2
C	a_3	b_3	c_3	d_3
D	a_4	b_4	c_4	d_4

The coordinates of the corners of the first tetrahedron are now known and the transference may be effected as before.

71. Ex. 1. If one corner as D be changed to D' without altering the opposite face show that the direction of the force at D' is parallel to the force at D, and that their magnitudes are inversely proportional to the distances of D and D' from the unchanged face. See the rule in Art. 60.

If D' lie in the plane BCD show that the force at A is unaltered.

Ex. 2. The forces at the corners of a tetrahedron $ABCD$ are F, G, H, L

respectively; it is required to find the central plane, the angles between the forces being given.

Let the cosine of the angle between two forces F, G be represented by $\cos FG$ and so on. Let f, g, h, l be the minors of the four constituents in the leading diagonal of the determinant.

$$\begin{vmatrix} 1 & \cos FG, & \cos FH, & \cos FL \\ \cos FG, & 1\ , & \cos GH, & \cos GL \\ \cos FH, & \cos GH, & 1 & \cos HL \\ \cos FL, & \cos GL, & \cos HL, & 1 \end{vmatrix}.$$

Then the central plane divides any side as AB in a point P such that

$$\frac{F \cdot AP}{G \cdot BP} = \pm \sqrt{\frac{f}{g}}.$$

Resolve the force F into three others F_1, F_2, F_3, acting parallel to G, H, L. Consider the three sets of parallel forces, viz. (G, F_1), (H, F_2), (L, F_3). We may collect each into its own centre of parallel forces and thus obtain three points on the central plane, Art. 58. The central plane therefore cuts AB in a point P where $F_1 \cdot AP = G \cdot BP$. But since F_1, F_2, F_3 are in equilibrium with $-F$, we have by Art. 48 of Vol. I, $F_1^2/F^2 = g/f$. The result follows at once.

72. If the forces F, G, H, of an astatic triangle ABC are rectangular and of finite magnitude, and if the area ABC is not zero, prove that the system cannot be reduced to fewer than three forces.

If possible let the forces be reduced to two, P and Q, and let these act at D and E in the plane of the triangle. Let p, q, r be the perpendiculars from A, B, C on DE. Turn the forces about their points of application until the force at A is perpendicular to the plane ABC, then the forces at B and C act in that plane. Taking moments about DE we have $Fp = 0$. Similarly $Gq = 0$, $Hr = 0$. But this is impossible if the area of the triangle is not zero.

That the points of application D, E must lie in the plane ABC follows from Art. 57, for DE may be regarded as one side of an astatic triangle, the third force being zero. We may also prove this in an elementary manner. Place the body so that the direction of the force P is parallel to the plane of ABC, while the other Q is not parallel; this is possible provided P and Q are not parallel to each other. Then, as in Art. 13, taking the plane of ABC as that of xy, we have Z_z the same for the three forces F, G, H and the two P, Q. The ordinate of E is therefore zero. In the same way the ordinate of D is zero.

If the forces P and Q are parallel to each other, they cannot form a couple because their components parallel to F, G and H are not zero. They can therefore be reduced to a single force. Proceeding as above we easily show that its point of application lies in the triangle; thence we deduce as before that the area of ABC is zero.

That the three forces F, G, H cannot be reduced to two, P, Q also follows from the invariants of an astatic triangle. Regarding DE as one side of the triangle, the third force being zero, we see that the second invariant of Art. 65 is zero. It follows that $FGH\Delta$ is also zero, which is impossible unless either the area Δ or one of the forces F, G, H is zero.

73. *To investigate the condition that the forces of an astatic system can be reduced to two forces.*

We have seen in Art. 56 that the forces of the system can be reduced to three forces, viz. X_0, Y_0, Z_0, acting at three points

A, B, C whose coordinates (x_1, y_1, z_1) (x_2, y_2, z_2) (x_3, y_3, z_3) are given by

$$\begin{array}{lll} X_0x_1 = X_x & X_0y_1 = X_y & X_0z_1 = X_z, \\ Y_0x_2 = Y_x & Y_0y_2 = Y_y & Y_0z_2 = Y_z, \\ Z_0x_3 = Z_x & Z_0y_3 = Z_y & Z_0z_3 = Z_z. \end{array}$$

We shall suppose in the first instance that the principal force is not zero, and that the axes are so chosen that X_0, Y_0, Z_0 are all finite.

If the three points A, B, C lie in a straight line we may make a further reduction. We can replace each of these forces by two other forces parallel to it and of proper magnitude, acting at any two points M_1, M_2, which lie in the straight line. By compounding the three forces at M_1, and also those at M_2, the whole system can be reduced to two forces. In order therefore that the system of forces may be reducible to two forces it is *sufficient* that the three points A, B, C should lie in a straight line.

It is also *necessary*, for otherwise the system is equivalent to an astatic triangle with rectangular forces. Now by Art. 72 such a system cannot be reduced to two forces unless either the triangle is evanescent or one at least of the forces X_0, Y_0, Z_0, is zero.

If the three points A, B, C lie in a straight line a plane can be drawn through that straight line and the origin. Hence

$$\begin{vmatrix} X_x, & X_y, & X_z \\ Y_x, & Y_y, & Y_z \\ Z_x, & Z_y, & Z_z \end{vmatrix} = 0.$$

The projection of these points on any coordinate plane must also lie in a straight line. We therefore have

$$\begin{vmatrix} X_0, & X_y, & X_z \\ Y_0, & Y_y, & Y_z \\ Z_0, & Z_y, & Z_z \end{vmatrix} = 0, \qquad \begin{vmatrix} X_x, & X_0, & X_z \\ Y_x, & Y_0, & Y_z \\ Z_x, & Z_0, & Z_z \end{vmatrix} = 0, \qquad \begin{vmatrix} X_x, & X_y, & X_0 \\ Y_x, & Y_y, & Y_0 \\ Z_x, & Z_y, & Z_0 \end{vmatrix} = 0.$$

The second of these four equations expresses the fact that A, B, C lie in a plane perpendicular to that of yz, the third that they lie in a plane perpendicular to that of xz, and so on.

Since no two of these four planes coincide, except when the points A, B, C lie in a coordinate plane, any two of the last three equations are sufficient to express the fact that the three points A, B, C lie in a straight line except when the three force components are zero.

These determinants are the coefficients of the several terms in

the equation to the central plane. That plane is therefore indeterminate.

Expressions for these determinants in terms of the forces, without the intervention of coordinate axes, have been given in Art. 31.

74. *To find the equivalent forces.* We have seen that they may be made to act at any two points M_1, M_2 which lie on the straight line ABC. The equation to this straight line is evidently $\dfrac{\xi - x_1}{x_2 - x_1} = \dfrac{\eta - y_1}{y_2 - y_1} = \dfrac{\zeta - z_1}{z_2 - z_1}$. This straight line is called the *central line of the two forces*.

If two forces, not parallel to each other, are together astatically equivalent to two other forces, we may prove in an elementary manner that the four points of application lie in one straight line.

Let P_1, P_2 acting at M_1, M_2 be equivalent to Q_1, Q_2 acting at N_1, N_2. Make P_1 act parallel to N_1N_2 and take moments about N_1N_2. It immediately follows that M_2 lies on N_1N_2. Similarly M_2 lies on N_1N_2. *Thus the central line is fixed in the body.*

Take any two distinct points M_1, M_2 on the central line. Let the coordinates of the points thus chosen be (f, g, h) and (f', g', h'). Let (F, G, H), (F', G', H') be the components of the forces at these two points. The forces will then be known when we have found (F, G, H) and (F', G', H').

Since this system of two forces is equivalent to the given system, the twelve elements must be the same for each system (Art. 12).

We therefore have

$$\begin{array}{llll} X_x = Ff + F'f', & X_y = Fg + F'g', & X_z = Fh + F'h', & X_0 = F + F' \\ Y_x = Gf + G'f', & Y_y = Gg + G'g', & Y_z = Gh + G'h', & Y_0 = G + G' \\ Z_x = Hf + H'f', & Z_y = Hg + H'g', & Z_z = Hh + H'h', & Z_0 = H + H'. \end{array}$$

Any six of these equations determine F, G, H; F', G', H' when f, g, h and f', g', h' are given.

75. *To shew that whatever points are chosen on the central line, the forces at those points are always parallel to the same plane.*

Supposing the system to be already reduced to two forces P_1, P_2 acting at some two points M_1, M_2, let us replace these by two other forces Q_1, Q_2 acting at any other points N_1, N_2 on the central line. The force Q_1 is the resultant of two forces which act parallel to P_1 and P_2; it is therefore parallel to any plane to which P_1 and P_2 are both parallel. In the same way the force Q_2 is parallel to the same plane.

It should also be noticed that the resultant of the two forces P_1, P_2, when transferred parallel to themselves to act at the same point, is a force fixed in direction and magnitude.

76. Referring to the determinantal conditions given in Art. 73, we see that *if we substitute ξ, η, ζ for the terms in any row* in the first determinant (repeated here in the margin) we have the equation of the plane containing the origin and the central line of the two resultant forces.

$$\begin{vmatrix} X_x, & X_y, & X_z \\ Y_x, & Y_y, & Y_z \\ Z_x, & Z_y, & Z_z \end{vmatrix} = 0$$

If however we substitute ξ, η, ζ for the terms in any column of the same determinantal equation, we have the equation of the plane to which the two resultant forces are parallel whatever be their points of application.

The first of these theorems follows at once from the values of x_1, &c. given in

Art. 73. The second is easily proved by substituting in the terms of the first and second columns the values of X_x &c. given in Art. 74, and in the third column ξ, η, ζ. After an obvious reduction and division by $fg'-f'g$, the equation reduces to the form shown in the margin, which is the plane required. There is no exceptional case when the divisor vanishes, for the equation to the plane then takes the form $0=0$.

$$\begin{vmatrix} F, & F', & \xi \\ G, & G', & \eta \\ H, & H', & \zeta \end{vmatrix} = 0$$

77. We have hitherto assumed that X_0, Y_0, Z_0 are all finite. The case in which any one or any two are zero may be treated as a limiting case and the corresponding conditions may be derived from those obtained when X_0, Y_0, Z_0 have finite but *general* values. As long as the conditions thus obtained are not nugatory they will be the conditions required. If however the principal force R is zero, the three components X_0, Y_0, Z_0 vanish for all axes and the reasoning in Art. 73 fails from the beginning.

The equations of Art. 74 supply a method of arriving at the conditions that the given forces can be reduced to two forces without making any assumption about the principal force. The body being in any position, let the components of the two forces be, as before (F, G, H), (F', G', H'), and let their points of application be (f, g, h), (f', g', h'). The required conditions may then be deduced from the twelve equations given in Art. 74. It is evident by simple inspection that the four determinantal equations given in Art. 73 are satisfied.

If the principal force is zero and the system can be reduced to two forces, those two forces must be equal and opposite, i.e. they must form a couple. Let $\pm F$, $\pm G$, $\pm H$ be the resolved parts of the forces of this couple, (f, g, h) (f', g', h') the coordinates of the extremities of its astatic arm. Then equating the nine finite elements of the system to those of the couple we have

$$\begin{aligned} X_x &= F(f'-f), & X_y &= F(g'-g), & X_z &= F(h'-h) \\ Y_x &= G(f'-f), & Y_y &= G(g'-g), & Y_z &= G(h'-h) \\ Z_x &= H(f'-f), & Z_z &= H(g'-g), & Z_z &= H(h'-h). \end{aligned}$$

The necessary and sufficient conditions that the system should be equivalent to two forces are therefore that (X_x, Y_x, Z_x), (X_y, Y_y, Z_y), (X_z, Y_z, Z_z), should be each proportional to the direction cosines of one straight line. This straight line is parallel to the forces of the couple.

78. Ex. 1. Show that any force F acting at a point A may be replaced by forces P_1, P_2 acting parallel to F at any two points M_1, M_2 such that AM_1M_2 is a straight line. Show also that these forces are

$$P_1 = F\frac{AM_2}{AM_2 - AM_1} \text{ and } P_2 = F\frac{AM_1}{AM_1 - AM_2}.$$

Ex. 2. Two given forces P_1, P_2, acting at the points M_1, M_2, are changed into two forces Q_1, Q_2 which are at right angles to each other, and act at two other points N_1, N_2 in the straight line M_1M_2. If y_1, y_2 are the distances of N_1, N_2 from the central point of the forces P_1, P_2, prove that $R^4y_1y_2 = -(P_1P_2D\sin\theta)^2$ where $R^2 = P_1^2 + P_2^2 + 2P_1P_2\cos\theta$, D is the distance M_1M_2 and θ is the inclination of the forces P_1, P_2 to each other. It follows that the product y_1y_2 is the same for all equivalent rectangular forces.

Ex. 3. In all transformations of two forces P_1, P_2 into two others in which the points of application remain on the same straight line, the quantities

(1) $P_1^2 + P_2^2 + 2P_1P_2\cos\theta$,

(2) $P_1P_2D\sin\theta$,

(3) $P_1(P_1 + P_2\cos\theta)x_1 + P_2(P_1\cos\theta + P_2)x_2$,

are invariable, where x_1, x_2 are the distances of the points of application M_1 from any fixed point on the central line, D is the distance M_1M_2 and θ is angle made by the forces with each other.

Ex. 4. A system consists of two forces P_1, P_2 acting at M_1, M_2 and the incl tion of the forces to each other is θ. Show that (1) the central point O is the ce of gravity of weights proportional to $P_1(P_1+P_2\cos\theta)$ and $P_2(P_1\cos\theta+P_2)$ place M_1, M_2. (2) The central ellipsoid at O is two parallel planes perpendicula M_1M_2. (3) The principal axes at O are M_1M_2 and any two perpendicular stra lines.

79. *To determine the conditions that the forces of an ast system reduce to a single force.*

Let the single force be P_1, let it act at the point $(x_1, y_1,$ and let its components be X_1, Y_1, Z_1. Comparing the elem at any base we have

$$X_x = X_1x_1, \qquad X_y = X_1y_1, \qquad X_z = X_1z_1, \text{ \&c.}$$

Hence we see that the constituents in any column of any of four determinants of Art. 73 bear to each other the ra (X_1, Y_1, Z_1) of the components of the single force and that t ratios must be the same for every column.

We also notice that the constituents in any row of any of four determinants bear to each other the ratios $(x_1, y_1, z_1$ $(1, y_1, z_1)$ &c. of the coordinates of the point of application.

We have twelve elementary equations and six arbit quantities (X_1, Y_1, Z_1), (x_1, y_1, z_1) leaving six conditions t satisfied by the elements of the system.

Since $X_0 = X_1$, &c., it is clear that the single equivalent is equal and parallel to the principal force, Art. 11. Also, the coordinates of the central point depend on the tw elements, it is evident that the central points of two equiv systems coincide, Art. 28. Thus it follows that the poi application of the equivalent single force is the central poi the system.

Note on Art. 106, Page 54.

The condition that a system of surfaces can be level.

The equation given in this article supplies us with a test by which we can determine whether any given system of surfaces can be level surfaces.

Let the surfaces be given by the equation

$$\phi(x, y, z, c) = 0 \quad\text{..........(1)},$$

where c is the parameter. If these are level surfaces, the parameter c, as given by this equation, is such that some function of c, say $f(c)$, satisfies Laplace's equation. Hence writing c for V, we have

$$-\frac{\dfrac{d^2f}{dc^2}}{\dfrac{df}{dc}} = \frac{\dfrac{d^2c}{dx^2} + \dfrac{d^2c}{dy^2} + \dfrac{d^2c}{dz^2}}{\left(\dfrac{dc}{dx}\right)^2 + \left(\dfrac{dc}{dy}\right)^2 + \left(\dfrac{dc}{dz}\right)^2} \quad\text{..........(2)}.$$

If the right-hand side of this equation is a function of c only, say $\psi(c)$, we have a differential equation to find $f(c)$. The result is

$$f(c) = A\int e^{-\int\psi(c)\,dc}\,dc + B \quad\text{..........(3)}.$$

The required test may be expressed in the following rule. Find the value of the right-hand side of (2) by differentiating (1) on the supposition that c is a function of x, y, z. If the resulting value can be expressed as a function of c only and not of x, y, z (by substitution from (i) if necessary), the given surfaces are level, but if not the surfaces are not level. This test is due to Lamé.

The function of c given by (3) when expressed in terms of x, y, z gives by simple differentiation the components of the forces corresponding to the level surfaces. It does not however follow that these forces can be produced by the attraction of any real finite body. Unless some portion of space is excluded from the system of surfaces it follows from Poisson's theorem that the attracting body must be at the points, lines, or surfaces at which the potential $f(c)$ is infinite.

As an example, let the given surfaces be prolate spheroids having their foci at two given points A and B. It may be shown that they can be level surfaces, and that the potential is infinite at every point of AB, see Art. 41.

Note on Art. 166, Page 88.

The invariant property of Laplace's functions.

We may notice that the invariant property of Laplace's functions, already referred to in Art. 78, reappears in this article in a slightly different form. Let any physical quantity, say the surface density of a thin stratum on the surface of a sphere, be represented by $\sigma = r^n Y_n$. Here σ is a homogeneous function of the coordinates x, y, z of a point P, and this function satisfies the equation $\nabla^2 V = 0$. Let us now transform the coordinate axes, keeping the origin unchanged, so that the coordinates of P become x', y', z'. Then by Art. 78, the equation $\nabla^2 V = 0$ retains the same form as before, except that x', y', z' has been written for x, y, z, and this equation must be satisfied by σ when expressed in terms of the new coordinates x', y', z'. Replacing these Cartesian axes by polars (r, θ, ϕ), (r, θ', ϕ')

having respectively the axes of z and z' for their axes of reference we have two equal expansions of Y_n referred to different axes, viz.

$$Y_n = a_0 P_n + \sin\theta \frac{dP_n}{d\mu}(a_1\cos\phi + b_1\sin\phi) + \&c. \quad\ldots\ldots\ldots\ldots(1),$$

$$= a_0' P_n' + \sin\theta' \frac{dP_n'}{d\mu'}(a_1'\cos\phi' + b_1'\sin\phi') + \&c. \quad\ldots\ldots\ldots\ldots(2),$$

where P_n, P_n' are the Legendre's functions corresponding to the different polar axes, and (θ, ϕ), (θ', ϕ') are the angular coordinates of the same line in the two systems of axes. The quantities a_0, a_1, b_1 &c., a_0', a_1', b_1' &c. are constants which depend on the system of axes chosen. Either set of constants may be found in terms of the other when the relations of the axes to each other are known. Thus, we may transform the Cartesian quantic $r^n Y_n$ by writing for x, y, z their values in terms of x', y', z'; or we may equate the two values of Y_n, given by the expressions (1) and (2), at different points of the sphere and thus obtain equations connecting the constants.

Note on Art. 19, Page 137.

Equation of the three moments.

In Prof. Cotterill's *Applied Mechanics*, 1884, page 337, it is stated that the theorem of the three moments was originally discovered by Clapeyron. An extension of the theorem to the case in which the external supports are on the same level, while the central one is at a small distance beneath that level, is also given in the same work.

Note on Art. 1, Page 165.

Moigno's Statique.

The theorems on Astatics given by Moigno may be found in his *Leçons de Mécanique Analytique*, 1868, which he tells us are chiefly founded on the methods of Cauchy. As his demonstrations are different from those given in this treatise, it may be useful to indicate the plan of his work, just as that of Darboux is given in the text.

First, by a transformation of axes, he obtains the twelve equations of equilibrium given in Art. 11. Thence he deduces the conditions that a system of forces can be astatically reduced to a single force by considering what single force can be in equilibrium with the system. Supposing these conditions not to be satisfied, he shows that the system can be reduced to two forces provided two conditions are satisfied. These conditions agree with the two last determinantal equations given in Art. 73. He next shows that the system can always be reduced to a force and two couples and that the point of application of the force may be arbitrarily chosen on a plane fixed in the body. This plane is defined to be the central plane. He then shows that if the arbitrary point is properly chosen the directions of the forces and of the arms may be simplified in the manner described in Art. 27. This point is defined to be the central point. Proceeding next to consider the case in which the body is so placed that the forces admit of a single resultant, he shows that that single resultant must intersect two conics fixed in the body. He next discusses the case in which the equilibrium is astatic only for displacements of the

body round a given axis; following the same plan as before, he enquires into the conditions that the system can be reduced to one, two or three forces. He concludes with an application to magnetic forces and investigates the positions of the central plane and central point.

NOTE ON ART 11, PAGE 171.

The conditions of equilibrium.

That the conditions of equilibrium given in Art. 11 are sufficient as well as necessary follows at once from the previous article. Thus, since the force F is the resultant of X_x/a, Y_x/a, Z_x/a, it is clear that F is zero. Similarly G and H are zero. Since X_0, Y_0, Z_0, are zero the principal force R is zero, so that the body is in equilibrium in all positions.

We may however also arrive at the same result independently. The body and forces in any one position being referred to axes x, y, z, let the twelve elements be zero. The axes x, y, z remaining fixed in space, let the body be moved about the origin into any other position, and let the coordinates of the point (x, y, z) become (x', y', z'). Since x, y, z are linear functions of x', y', z' whose coefficients are independent of the coordinates, it is evident that the twelve elements $\Sigma Xx'$ &c. are also zero. The six statical equations of equilibrium referred to in Art. 11 are therefore satisfied in this new position of the body.

NOTE ON ART. 33, PAGE 183.

Analogy of astatics to moments of inertia.

The analogy may be made more distinct by adding another proposition to those given in the text.

Let O be the central point, Oy, Oz the principal astatic axes in the central plane, Ox that perpendicular. The astatic moment K about any axis OP, whose direction cosines are l, m, n is given by

$$K^2 = K_2^2 m^2 + K_3^2 n^2 \quad\quad (1).$$

Let a lamina be placed in the plane of yz with its centre of gravity at O, having the axes of x, y, z for its principal axes of inertia; and let K_2^2, K_3^2 be its moments of inertia at the origin with regard to the planes respectively perpendicular to the axes of y and z. The equation (1) then shows that K^2 is the moment of inertia of the lamina with regard to a plane drawn through O perpendicular to OP.

Let O' be any other point whose coordinates are ξ, η, ζ, and let $O'P'$ be parallel to OP. The astatic moment K' at O' corresponding to the arm $O'P'$ is given by

$$K'^2 = K^2 + R^2 p^2 \quad\quad (2),$$

where p is the projection of OO' on OP. This is also the formula which gives the moment of inertia of the lamina with regard to a plane drawn through O' perpendicular to $O'P'$, provided R^2 is the mass of the body.

It follows that the moment of inertia of the lamina with regard to a plane drawn through any point O' perpendicular to any straight line $O'P'$ represents the square of the astatic moment at the base O' for the arm $O'P'$.

Since the moments of inertia for all arms through O' represent the squares of the astatic moments for the same arms, it follows that they have the same maxima

and minima and are connected together by the same rules. The principal axes of inertia at O' are therefore the same in direction as the principal astatic axes at O'.

That the principal astatic moments at O' are the normals to the confocals (4) of Art. 34, and that the astatic moments are the three values of M given by the cubic, follow at once from the properties of the principal axes of inertia, see *Rigid Dynamics*, Vol. I. Art. 56.

Since the moments of inertia of the lamina about the axes of y and z are respectively K_3^2 and K_2^2, it follows that the lamina might take the form of a homogeneous elliptic disc, whose semi-axes of y and z are respectively $2K_2/R$ and $2K_3/R$, and whose mass is R^2. The boundary is therefore similar to the imaginary focal conic.

INDEX TO ATTRACTIONS.

The numbers refer to the articles.

INDEX TO THE BENDING OF RODS.

INDEX TO ASTATICS.

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